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Midpoint algebras

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Summary. In this article basic properties of midpoint algebras are proved. We define a congruence relation $\equiv$ on bound vectors and free vectors as the equivalence classes of $\equiv$.

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The notation and terminology used in this paper are introduced in the following articles: [5], [1], [2], [3], [4], and [6]. We consider midpoint algebra structures which are systems

$\langle$ points, a midpoint operation $\rangle$

where the points is a non-empty set and the midpoint operation is a binary operation on the points. In the sequel MS is a midpoint algebra structure and a, b are elements of the points of MS. Let us consider MS, a, b. The functor $a \oplus b$ yielding an element of the points of MS, is defined by:

$a \oplus b =$ (the midpoint operation of $MS)(a, b)$.

We now state a proposition

(1) $a \oplus b =$ (the midpoint operation of $MS)(a, b)$.

Let x be arbitrary. Then $\{x\}$ is a non-empty set.

zo is a binary operation on $\{0\}$.

One can prove the following propositions:

(2) zo is a function from $[\!:\{0\}, \{0\}:\!]$ into $\{0\}$.

(3) For all elements x, y of $\{0\}$ holds $zo(x, y) = 0$.

The midpoint algebra structure EX is defined by:

$\mathrm{EX} = \langle \{0\}, zo \rangle$.

The following propositions are true:

(4) $\mathrm{EX} = \langle \{0\}, zo \rangle$.

(5) The points of $\mathrm{EX} = \{0\}$.

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(6) The midpoint operation of $\mathrm{EX} = zo$.

(7) For every element a of the points of EX holds $a = 0$.

(8) For all elements a, b of the points of EX holds $a \oplus b = zo(a, b)$.

(9) For all elements a, b of the points of EX holds $a \oplus b = 0$.

(10) For all elements a, b, c, d of the points of EX holds $a \oplus a = a$ and $a \oplus b = b \oplus a$ and $(a \oplus b) \oplus (c \oplus d) = (a \oplus c) \oplus (b \oplus d)$ and there exists an element x of the points of EX such that $x \oplus a = b$.

A midpoint algebra structure is called a midpoint algebra if:

for all elements a, b, c, d of the points of it holds $a \oplus a = a$ and $a \oplus b = b \oplus a$ and $(a \oplus b) \oplus (c \oplus d) = (a \oplus c) \oplus (b \oplus d)$ and there exists an element x of the points of it such that $x \oplus a = b$.

We follow the rules: M denotes a midpoint algebra and a, b, c, d, a', b', c', d', x, y, x' denote elements of the points of M. Next we state several propositions:

(11) $a \oplus a = a$.

(12) $a \oplus b = b \oplus a$.

(13) $(a \oplus b) \oplus (c \oplus d) = (a \oplus c) \oplus (b \oplus d)$.

(14) There exists x such that $x \oplus a = b$.

(15) $(a \oplus b) \oplus c = (a \oplus c) \oplus (b \oplus c)$.

(16) $a \oplus (b \oplus c) = (a \oplus b) \oplus (a \oplus c)$.

(17) If $a \oplus b = a$, then $a = b$.

(18) If $x \oplus a = x' \oplus a$, then $x = x'$.

(19) If $a \oplus x = a \oplus x'$, then $x = x'$.

Let us consider M, a, b, c, d. The predicate $a, b \equiv c, d$ is defined by:

$a \oplus d = b \oplus c$.

The following propositions are true:

(20) $a, b \equiv c, d$ if and only if $a \oplus d = b \oplus c$.

(21) $a, a \equiv b, b$.

(22) If $a, b \equiv c, d$, then $c, d \equiv a, b$.

(23) If $a, a \equiv b, c$, then $b = c$.

(24) If $a, b \equiv c, c$, then $a = b$.

(25) $a, b \equiv a, b$.

(26) There exists d such that $a, b \equiv c, d$.

(27) If $a, b \equiv c, d$ and $a, b \equiv c, d'$, then $d = d'$.

(28) If $x, y \equiv a, b$ and $x, y \equiv c, d$, then $a, b \equiv c, d$.

(29) If $a, b \equiv a', b'$ and $b, c \equiv b', c'$, then $a, c \equiv a', c'$.

In the sequel p, q, r will denote elements of $[\!:$ the points of M, the points of $M :\!]$. Let us consider M, p. Then $p_{\mathbf{1}}$ is an element of the points of M.

Let us consider M, p. Then $p_{\mathbf{2}}$ is an element of the points of M.

Let us consider M, p, q. The predicate $p \equiv q$ is defined as follows:

$p_{\mathbf{1}}, p_{\mathbf{2}} \equiv q_{\mathbf{1}}, q_{\mathbf{2}}$.

One can prove the following proposition

(30) $p \equiv q$ if and only if $p_{\mathbf{1}}, p_{\mathbf{2}} \equiv q_{\mathbf{1}}, q_{\mathbf{2}}$.

Let us consider M, a, b. Then $\langle a, b\rangle$ is an element of $[\!:$ the points of M, the points of $M :\!]$.

One can prove the following propositions:

(31) If $a, b \equiv c, d$, then $\langle a, b\rangle \equiv \langle c, d\rangle$.

(32) If $\langle a, b\rangle \equiv \langle c, d\rangle$, then $a, b \equiv c, d$.

(33) $p \equiv p$.

(34) If $p \equiv q$, then $q \equiv p$.

(35) If $p \equiv q$ and $p \equiv r$, then $q \equiv r$.

(36) If $p \equiv r$ and $q \equiv r$, then $p \equiv q$.

(37) If $p \equiv q$ and $q \equiv r$, then $p \equiv r$.

(38) If $p \equiv q$, then $r \equiv p$ if and only if $r \equiv q$.

(39) For every p holds $\{q : q \equiv p\}$is a non-empty subset of $[\!:$ the points of M, the points of $M :\!]$.

Let us consider M, p. The functor $p^{\smile}$ yields a non-empty subset of $[\!:$ the points of M, the points of $M :\!]$ and is defined as follows:

$p^{\smile} = \{q : q \equiv p\}$.

The following propositions are true:

(40) For every p holds $p^{\smile} = \{q : q \equiv p\}$ and $p^{\smile}$ is a non-empty subset of $[\!:$ the points of M, the points of $M :\!]$.

(41) For every p holds $r \in p^{\smile}$ if and only if $r \equiv p$.

(42) If $p \equiv q$, then $p^{\smile} = q^{\smile}$.

(43) If $p^{\smile} = q^{\smile}$, then $p \equiv q$.

(44) If $\langle a, b\rangle^{\smile} = \langle c, d\rangle^{\smile}$, then $a \oplus d = b \oplus c$.

(45) $p \in p^{\smile}$.

Let us consider M. A non-empty subset of $[\!:$ the points of M, the points of $M :\!]$ is said to be a vector of M if:

there exists p such that it $= p^{\smile}$.

The following proposition is true

(46) For every non-empty subset X of $[\!:$ the points of M, the points of $M :\!]$ holds X is a vector of M if and only if there exists p such that $X = p^{\smile}$.

In the sequel u, v, w, w' denote vectors of M. The following proposition is true

(47) $p^{\smile}$ is a vector of M.

Let us consider M, p. Then $p^{\smile}$ is a vector of M.

We now state a proposition

(48) There exists u such that for every p holds $p \in u$ if and only if $p_{\mathbf{1}} = p_{\mathbf{2}}$.

Let us consider M. The functor I_M yielding a vector of M, is defined by:

$\mathrm{I}_M = \{p : p_{\mathbf{1}} = p_{\mathbf{2}}\}$.

Next we state four propositions:

(49) $\mathrm{I}_M = \{p : p_{\mathbf{1}} = p_{\mathbf{2}}\}$.

(50) $\mathrm{I}_M = \langle b, b \rangle^{\smile}$.

(51) There exist w, p, q such that $u = p^{\smile}$ and $v = q^{\smile}$ and $p_{\mathbf{2}} = q_{\mathbf{1}}$ and $w = \langle p_{\mathbf{1}}, q_{\mathbf{2}} \rangle^{\smile}$.

(52) Suppose that

(i) there exist p, q such that $u = p^{\smile}$ and $v = q^{\smile}$ and $p_{\mathbf{2}} = q_{\mathbf{1}}$ and $w = \langle p_{\mathbf{1}}, q_{\mathbf{2}} \rangle^{\smile}$,

(ii) there exist p, q such that $u = p^{\smile}$ and $v = q^{\smile}$ and $p_{\mathbf{2}} = q_{\mathbf{1}}$ and $w' = \langle p_{\mathbf{1}}, q_{\mathbf{2}} \rangle^{\smile}$.

Then $w = w'$.

Let us consider M, u, v. The functor $u+v$ yields a vector of M and is defined by:

there exist p, q such that $u = p^{\smile}$ and $v = q^{\smile}$ and $p_{\mathbf{2}} = q_{\mathbf{1}}$ and $u + v = \langle p_{\mathbf{1}}, q_{\mathbf{2}} \rangle^{\smile}$.

We now state a proposition

(53) There exists b such that $u = \langle a, b \rangle^{\smile}$.

Let us consider M, a, b. The functor $\overrightarrow{[a, b]}$ yields a vector of M and is defined by:

$\overrightarrow{[a, b]} = \langle a, b \rangle^{\smile}$.

Next we state a number of propositions:

(54) $\overrightarrow{[a, b]} = \langle a, b \rangle^{\smile}$.

(55) There exists b such that $u = \overrightarrow{[a, b]}$.

(56) If $\langle a, b \rangle \equiv \langle c, d \rangle$, then $\overrightarrow{[a, b]} = \overrightarrow{[c, d]}$.

(57) If $\overrightarrow{[a, b]} = \overrightarrow{[c, d]}$, then $a \oplus d = b \oplus c$.

(58) $\mathrm{I}_M = \overrightarrow{[b, b]}$.

(59) If $\overrightarrow{[a, b]} = \overrightarrow{[a, c]}$, then $b = c$.

(60) $\overrightarrow{[a, b]} + \overrightarrow{[b, c]} = \overrightarrow{[a, c]}$.

(61) $\langle a, a \oplus b \rangle \equiv \langle a \oplus b, b \rangle$.

(62) $\overrightarrow{[a, a \oplus b]} + \overrightarrow{[a, a \oplus b]} = \overrightarrow{[a, b]}$.

(63) $(u + v) + w = u + (v + w)$.

(64) $u + \mathrm{I}_M = u$.

(65) There exists v such that $u + v = \mathrm{I}_M$.

(66) $u + v = v + u$.

(67) If $u + v = u + w$, then $v = w$.

Let us consider M, u. The functor $-u$ yields a vector of M and is defined by:

$u + (-u) = \mathrm{I}_M$.

We now state a proposition

(68) $u + (-u) = \mathrm{I}_M$.

In the sequel X denotes an element of $2^{[\text{the points of } M, \text{the points of } M]}$. Let us consider M. The functor setvect M yields a set and is defined as follows:

setvect $M = \{X : X \text{ is a vector of } M\}$.

Next we state a proposition

(69) setvect $M = \{X : X \text{ is a vector of } M\}$.

In the sequel x is arbitrary. One can prove the following two propositions:

(70) u is an element of $2^{[\text{the points of } M, \text{the points of } M]}$.

(71) x is a vector of M if and only if $x \in$ setvect M.

Let us consider M. Then setvect M is a non-empty set.

The following proposition is true

(72) x is a vector of M if and only if x is an element of setvect M.

In the sequel u_1, v_1, w_1, W, W_1, W_2, T will denote elements of setvect M. Let us consider M, u_1, v_1. The functor $u_1 + v_1$ yields an element of setvect M and is defined as follows:

for all u, v such that $u_1 = u$ and $v_1 = v$ holds $u_1 + v_1 = u + v$.

One can prove the following propositions:

(73) If $u_1 = u$ and $v_1 = v$, then $u_1 + v_1 = u + v$.

(74) $u_1 + v_1 = v_1 + u_1$.

(75) $(u_1 + v_1) + w_1 = u_1 + (v_1 + w_1)$.

Let us consider M. The functor addvect M yields a binary operation on setvect M and is defined as follows:

for all u_1, v_1 holds $(\text{addvect } M)(u_1, v_1) = u_1 + v_1$.

The following three propositions are true:

(76) $(\text{addvect } M)(u_1, v_1) = u_1 + v_1$.

(77) For every W there exists T such that $W + T = \mathrm{I}_M$.

(78) For all W, W_1, W_2 such that $W + W_1 = \mathrm{I}_M$ and $W + W_2 = \mathrm{I}_M$ holds $W_1 = W_2$.

Let us consider M. The functor complvect M yielding a unary operation on setvect M, is defined by:

for every W holds $W + (\text{complvect } M)(W) = \mathrm{I}_M$.

One can prove the following proposition

(79) $W + (\text{complvect } M)(W) = \mathrm{I}_M$.

Let us consider M. The functor zerovect M yields an element of setvect M and is defined as follows:

zerovect $M = \mathrm{I}_M$.

The following proposition is true

(80) zerovect $M = \mathrm{I}_M$.

Let us consider M. The functor vectgroup M yielding a group structure, is defined by:

vectgroup $M = \langle$setvect M, addvect M, complvect M, zerovect $M\rangle$.

Next we state several propositions:

(81) vectgroup $M = \langle$setvect M, addvect M, complvect M, zerovect $M\rangle$.

(82) The carrier of vectgroup $M =$ setvect M.

(83) The addition of vectgroup $M =$ addvect M.

(84) The reverse-map of vectgroup $M =$ complvect M.

(85) The zero of vectgroup $M =$ zerovect M.

(86) vectgroup M is an Abelian group.

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