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Real Functions Spaces [1]

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Summary. This abstract contains a construction of the domain of functions defined in an arbitrary nonempty set, with values being real numbers. In every such set of functions we introduce several algebraic operations, which yield in this set the structures of a real linear space, of a ring, and of a real algebra. Formal definitions of such concepts are given.

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The notation and terminology used in this paper are introduced in the following papers: [3], [9], [11], [2], [7], [12], [6], [1], [10], [4], [5], and [8]. We adopt the following convention: x_1, x_2, z are arbitrary and A, B denote non-empty sets. Let us consider A, B, and let F be a binary operation on B^A, and let f, g be elements of B^A. Then $F(f,g)$ is an element of B^A.

Let A, B, C, D be non-empty sets, and let F be a function from $[\!: C, D :\!]$ into B^A, and let cd be an element of $[\!: C, D :\!]$. Then $F(cd)$ is an element of B^A.

Let A, B be non-empty sets, and let f be a function from A into B. The functor $@f$ yields an element of B^A and is defined by:

$@f = f$.

We now state a proposition

(1) For all functions f, g from A into B holds $@f = g$ if and only if $f = g$.

In the sequel f, g, h denote elements of $\mathbb{R}^A$. Let A, B be non-empty sets, and let x be an element of B^A. The functor $\downharpoonleft x$ yields an element of B^A **qua** a non-empty set and is defined as follows:

$\downharpoonleft x = x$.

We now state a proposition

(2) For all elements f, g of B^A holds $\downharpoonleft f = g$ if and only if $f = g$.

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Let us consider A, B, and let f be an element of B^A **qua** a non-empty set. The functor $f{\downharpoonright}$ yielding an element of B^A, is defined by:

$f{\downharpoonright} = f$.

We now state two propositions:

(3) For all elements f, g of B^A **qua** non-empty sets holds $f{\downharpoonright} = g$ if and only if $f = g$.

(4) $f = ({\downharpoonright} f){\downharpoonright}$.

Let X, Z be non-empty sets, and let F be a binary operation on X, and let f, g be functions from Z into X. Then $F^{\circ}(f, g)$ is an element of X^Z.

Let X, Z be non-empty sets, and let F be a binary operation on X, and let a be an element of X, and let f be a function from Z into X. Then $F^{\circ}(a, f)$ is an element of X^Z.

Let us consider A. The functor $+_{\mathbb{R}^A}$ yields a binary operation on $\mathbb{R}^A$ and is defined by:

for all elements f, g of $\mathbb{R}^A$ holds $+_{\mathbb{R}^A}(f, g) = +_{\mathbb{R}}{}^{\circ}(f, g)$.

We now state a proposition

(5) For every binary operation F on $\mathbb{R}^A$ holds $F = +_{\mathbb{R}^A}$ if and only if for all elements f, g of $\mathbb{R}^A$ holds $F(f, g) = +_{\mathbb{R}}{}^{\circ}(f, g)$.

Let us consider A. The functor $\cdot_{\mathbb{R}^A}$ yields a binary operation on $\mathbb{R}^A$ and is defined as follows:

for all elements f, g of $\mathbb{R}^A$ holds $\cdot_{\mathbb{R}^A}(f, g) = \cdot_{\mathbb{R}}{}^{\circ}(f, g)$.

Next we state a proposition

(6) For every binary operation F on $\mathbb{R}^A$ holds $F = \cdot_{\mathbb{R}^A}$ if and only if for all elements f, g of $\mathbb{R}^A$ holds $F(f, g) = \cdot_{\mathbb{R}}{}^{\circ}(f, g)$.

Let us consider A, and let a be a real number, and let f be an element of $\mathbb{R}^A$. Then $\langle a, f \rangle$ is an element of $[\!:\mathbb{R}, \mathbb{R}^A:\!]$.

Let us consider A. The functor $\cdot^{\mathbb{R}}_{\mathbb{R}^A}$ yielding a function from $[\!:\mathbb{R}, \mathbb{R}^A:\!]$ into $\mathbb{R}^A$, is defined as follows:

for every real number a and for every element f of $\mathbb{R}^A$ and for every element x of A holds $(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle))(x) = a \cdot f(x)$.

The following proposition is true

(7) For every function F from $[\!:\mathbb{R}, \mathbb{R}^A:\!]$ into $\mathbb{R}^A$ holds $F = \cdot^{\mathbb{R}}_{\mathbb{R}^A}$ if and only if for every real number a and for every element f of $\mathbb{R}^A$ and for every element x of A holds $(F(\langle a, f \rangle))(x) = a \cdot f(x)$.

Let us consider A. The functor $\mathbf{0}_{\mathbb{R}^A}$ yields an element of $\mathbb{R}^A$ and is defined by:

$\mathbf{0}_{\mathbb{R}^A} = A \longmapsto 0$.

The following proposition is true

(8) For every element f of $\mathbb{R}^A$ holds $f = \mathbf{0}_{\mathbb{R}^A}$ if and only if $f = A \longmapsto 0$.

Let us consider A. The functor $\mathbf{1}_{\mathbb{R}^A}$ yields an element of $\mathbb{R}^A$ and is defined by:

$\mathbf{1}_{\mathbb{R}^A} = A \longmapsto 1$.

We now state several propositions:

(9) For every element f of $\mathbb{R}^A$ holds $f = \mathbf{1}_{\mathbb{R}^A}$ if and only if $f = A \longmapsto 1$.

(10) $h = +_{\mathbb{R}^A}(f, g)$ if and only if for every element x of A holds $h(x) = f(x) + g(x)$.

(11) $h = \cdot_{\mathbb{R}^A}(f, g)$ if and only if for every element x of A holds $h(x) = f(x) \cdot g(x)$.

(12) For every element x of A holds $\mathbf{1}_{\mathbb{R}^A}(x) = 1$.

(13) For every element x of A holds $\mathbf{0}_{\mathbb{R}^A}(x) = 0$.

(14) $\mathbf{0}_{\mathbb{R}^A} \neq \mathbf{1}_{\mathbb{R}^A}$.

In the sequel a, b are real numbers. The following proposition is true

(15) $h = \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle)$ if and only if for every element x of A holds $h(x) = a \cdot f(x)$.

One can prove the following propositions:

(16) $+_{\mathbb{R}^A}(f, g) = +_{\mathbb{R}^A}(g, f)$.

(17) $+_{\mathbb{R}^A}(f, +_{\mathbb{R}^A}(g, h)) = +_{\mathbb{R}^A}(+_{\mathbb{R}^A}(f, g), h)$.

(18) $\cdot_{\mathbb{R}^A}(f, g) = \cdot_{\mathbb{R}^A}(g, f)$.

(19) $\cdot_{\mathbb{R}^A}(f, \cdot_{\mathbb{R}^A}(g, h)) = \cdot_{\mathbb{R}^A}(\cdot_{\mathbb{R}^A}(f, g), h)$.

(20) $\cdot_{\mathbb{R}^A}(\mathbf{1}_{\mathbb{R}^A}, f) = f$.

(21) $+_{\mathbb{R}^A}(\mathbf{0}_{\mathbb{R}^A}, f) = f$.

(22) $+_{\mathbb{R}^A}(f, \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle -1, f \rangle)) = \mathbf{0}_{\mathbb{R}^A}$.

(23) $\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle 1, f \rangle) = f$.

(24) $\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, f \rangle) \rangle) = \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a \cdot b, f \rangle)$.

(25) $+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, f \rangle)) = \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a + b, f \rangle)$.

(26) $\cdot_{\mathbb{R}^A}(f, +_{\mathbb{R}^A}(g, h)) = +_{\mathbb{R}^A}(\cdot_{\mathbb{R}^A}(f, g), \cdot_{\mathbb{R}^A}(f, h))$.

(27) $\cdot_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle), g) = \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, \cdot_{\mathbb{R}^A}(f, g) \rangle)$.

(28) Suppose $x_1 \in A$ and $x_2 \in A$ and $x_1 \neq x_2$. Then there exist f, g such that for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ but if $z \neq x_1$, then $f(z) = 0$ and for every z such that $z \in A$ holds if $z = x_1$, then $g(z) = 0$ but if $z \neq x_1$, then $g(z) = 1$.

(29) Suppose that

(i) $x_1 \in A$,

(ii) $x_2 \in A$,

(iii) $x_1 \neq x_2$,

(iv) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ but if $z \neq x_1$, then $f(z) = 0$,

(v) for every z such that $z \in A$ holds if $z = x_1$, then $g(z) = 0$ but if $z \neq x_1$, then $g(z) = 1$.

Then for all a, b such that $+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g \rangle)) = \mathbf{0}_{\mathbb{R}^A}$ holds $a = 0$ and $b = 0$.

(30) If $x_1 \in A$ and $x_2 \in A$ and $x_1 \neq x_2$, then there exist f, g such that for all a, b such that $+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g \rangle)) = \mathbf{0}_{\mathbb{R}^A}$ holds $a = 0$ and $b = 0$.

(31) Suppose that
(i) $A = \{x_1, x_2\}$,
(ii) $x_1 \neq x_2$,
(iii) for every z such that $z \in A$ holds if $z = x_1$, then $f(z) = 1$ but if $z \neq x_1$, then $f(z) = 0$,
(iv) for every z such that $z \in A$ holds if $z = x_1$, then $g(z) = 0$ but if $z \neq x_1$, then $g(z) = 1$.
Then for every h there exist a, b such that
$h = +_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g \rangle))$.

(32) If $A = \{x_1, x_2\}$ and $x_1 \neq x_2$, then there exist f, g such that for every h there exist a, b such that $h = +_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g \rangle))$.

(33) Suppose $A = \{x_1, x_2\}$ and $x_1 \neq x_2$. Then there exist f, g such that for all a, b such that $+_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g \rangle)) = \mathbf{0}_{\mathbb{R}^A}$ holds $a = 0$ and $b = 0$ and for every h there exist a, b such that $h = +_{\mathbb{R}^A}(\cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle a, f \rangle), \cdot^{\mathbb{R}}_{\mathbb{R}^A}(\langle b, g \rangle))$.

(34) $\langle \mathbb{R}^A, \downharpoonleft \mathbf{0}_{\mathbb{R}^A}, +_{\mathbb{R}^A}, \cdot^{\mathbb{R}}_{\mathbb{R}^A} \rangle$ is a real linear space.

Let us consider A. The functor $\mathbb{R}^A_{\mathbb{R}}$ yields a real linear space and is defined by:

$\mathbb{R}^A_{\mathbb{R}} = \langle \mathbb{R}^A, \downharpoonleft \mathbf{0}_{\mathbb{R}^A}, +_{\mathbb{R}^A}, \cdot^{\mathbb{R}}_{\mathbb{R}^A} \rangle$.

We now state two propositions:

(35) $\mathbb{R}^A_{\mathbb{R}} = \langle \mathbb{R}^A, \downharpoonleft \mathbf{0}_{\mathbb{R}^A}, +_{\mathbb{R}^A}, \cdot^{\mathbb{R}}_{\mathbb{R}^A} \rangle$.

(36) $\mathbb{R}^A_{\mathbb{R}}$ is a real linear space.

In the sequel V will denote a real linear space and u, v, w will denote vectors of V. The following proposition is true

(37) There exists V and there exist u, v such that for all a, b such that $a \cdot u + b \cdot v = 0_V$ holds $a = 0$ and $b = 0$ and for every w there exist a, b such that $w = a \cdot u + b \cdot v$.

We consider ring structures which are systems

$\langle$ a carrier, a multiplication, an addition, a unity, a zero $\rangle$

where the carrier is a non-empty set, the multiplication, the addition are binary operations on the carrier, and the unity, the zero are elements of the carrier. In the sequel RS will be a ring structure. We now define four new functors. Let us consider RS. The functor 1_{RS} yields an element of the carrier of RS and is defined as follows:

$1_{RS} =$ the unity of RS.

The functor 0_{RS} yields an element of the carrier of RS and is defined as follows:

$0_{RS} =$ the zero of RS.

Let x, y be elements of the carrier of RS. The functor $x \cdot y$ yielding an element of the carrier of RS, is defined by:

$x \cdot y =$ (the multiplication of RS)(x, y).

The functor $x + y$ yielding an element of the carrier of RS, is defined by:

$x + y =$ (the addition of RS)(x, y).

In the sequel x, y denote elements of the carrier of RS. One can prove the following four propositions:

(38) (the multiplication of $RS)(x, y) = x \cdot y$.

(39) (the addition of $RS)(x, y) = x + y$.

(40) $1_{RS} =$ the unity of RS.

(41) $0_{RS} =$ the zero of RS.

Let us consider A. The functor $\mathrm{RRing}\, A$ yielding a ring structure, is defined by:

$\mathrm{RRing}\, A = \langle \mathbb{R}^A, \cdot_{\mathbb{R}^A}, +_{\mathbb{R}^A}, \rfloor \mathbf{1}_{\mathbb{R}^A}, \rfloor \mathbf{0}_{\mathbb{R}^A} \rangle$.

Next we state a proposition

(42) Let x, y, z be elements of the carrier of $\mathrm{RRing}\, A$. Then

(i) $x + y = y + x$,
(ii) $(x + y) + z = x + (y + z)$,
(iii) $x + 0_{\mathrm{RRing}\, A} = x$,
(iv) there exists an element t of the carrier of $\mathrm{RRing}\, A$ such that $x + t = 0_{\mathrm{RRing}\, A}$,
(v) $x \cdot y = y \cdot x$,
(vi) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
(vii) $x \cdot (1_{\mathrm{RRing}\, A}) = x$,
(viii) $x \cdot (y + z) = x \cdot y + x \cdot z$.

A ring structure is said to be a ring if:

Let x, y, z be elements of the carrier of it . Then

(i) $x + y = y + x$,
(ii) $(x + y) + z = x + (y + z)$,
(iii) $x + 0_{\mathrm{it}} = x$,
(iv) there exists an element t of the carrier of it such that $x + t = 0_{\mathrm{it}}$,
(v) $x \cdot y = y \cdot x$,
(vi) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
(vii) $x \cdot (1_{\mathrm{it}}) = x$,
(viii) $x \cdot (y + z) = x \cdot y + x \cdot z$.

One can prove the following proposition

(43) $\mathrm{RRing}\, A$ is a ring.

We consider algebra structures which are systems

$\langle$ a carrier, a multiplication, an addition, a multiplication$_1$, a unity, a zero $\rangle$

where the carrier is a non-empty set, the multiplication, the addition are binary operations on the carrier, the multiplication$_1$ is a function from $[\!:\mathbb{R},$ the carrier$:\!]$ into the carrier, and the unity, the zero are elements of the carrier. In the sequel AlS denotes an algebra structure. We now define four new functors. Let us consider AlS. The functor 1_{AlS} yielding an element of the carrier of AlS, is defined as follows:

$1_{AlS} =$ the unity of AlS.

The functor 0_{AlS} yielding an element of the carrier of AlS, is defined by:

$0_{AlS} =$ the zero of AlS.

Let x, y be elements of the carrier of AlS. The functor $x \cdot y$ yields an element of the carrier of AlS and is defined by:

$x \cdot y =$ (the multiplication of AlS)(x, y).

The functor $x+y$ yielding an element of the carrier of AlS, is defined as follows:

$x + y =$ (the addition of AlS)(x, y).

Let us consider AlS, and let x be an element of the carrier of AlS, and let a be a real number. The functor $a \cdot x$ yields an element of the carrier of AlS and is defined as follows:

$a \cdot x =$ (the multiplication$_1$ of AlS)$(\langle a, x \rangle)$.

In the sequel x, y are elements of the carrier of AlS. Next we state several propositions:

(44) (the multiplication of AlS)$(x, y) = x \cdot y$.

(45) (the addition of AlS)$(x, y) = x + y$.

(46) (the multiplication$_1$ of AlS)$(\langle a, x \rangle) = a \cdot x$.

(47) $0_{AlS} =$ the zero of AlS.

(48) $1_{AlS} =$ the unity of AlS.

Let us consider A. The functor RAlgebra A yielding an algebra structure, is defined as follows:

RAlgebra $A = \langle \mathbb{R}^A, \cdot_{\mathbb{R}^A}, +_{\mathbb{R}^A}, \cdot^{\mathbb{R}}_{\mathbb{R}^A}, \downharpoonleft \mathbf{1}_{\mathbb{R}^A}, \downharpoonleft \mathbf{0}_{\mathbb{R}^A} \rangle$.

The following proposition is true

(49) Let x, y, z be elements of the carrier of RAlgebra A. Given a, b. Then

- (i) $x + y = y + x$,
- (ii) $(x + y) + z = x + (y + z)$,
- (iii) $x + 0_{\mathrm{RAlgebra}\,A} = x$,
- (iv) there exists an element t of the carrier of RAlgebra A such that $x + t = 0_{\mathrm{RAlgebra}\,A}$,
- (v) $x \cdot y = y \cdot x$,
- (vi) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
- (vii) $x \cdot (1_{\mathrm{RAlgebra}\,A}) = x$,
- (viii) $x \cdot (y + z) = x \cdot y + x \cdot z$,
- (ix) $a \cdot (x \cdot y) = (a \cdot x) \cdot y$,
- (x) $a \cdot (x + y) = a \cdot x + a \cdot y$,
- (xi) $(a + b) \cdot x = a \cdot x + b \cdot x$,
- (xii) $(a \cdot b) \cdot x = a \cdot (b \cdot x)$.

An algebra structure is said to be an algebra if:

Let x, y, z be elements of the carrier of it . Given a, b. Then

- (i) $x + y = y + x$,
- (ii) $(x + y) + z = x + (y + z)$,
- (iii) $x + 0_{\mathrm{it}} = x$,
- (iv) there exists an element t of the carrier of it such that $x + t = 0_{\mathrm{it}}$,
- (v) $x \cdot y = y \cdot x$,
- (vi) $(x \cdot y) \cdot z = x \cdot (y \cdot z)$,
- (vii) $x \cdot (1_{\mathrm{it}}) = x$,
- (viii) $x \cdot (y + z) = x \cdot y + x \cdot z$,
- (ix) $a \cdot (x \cdot y) = (a \cdot x) \cdot y$,

(x) $a \cdot (x + y) = a \cdot x + a \cdot y$,
(xi) $(a + b) \cdot x = a \cdot x + b \cdot x$,
(xii) $(a \cdot b) \cdot x = a \cdot (b \cdot x)$.

The following proposition is true

(50) RAlgebra A is an algebra.

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