FORMALIZED MATHEMATICS
Vol.1, No.3, May–August 1990
Université Catholique de Louvain

Finite Sequences and Tuples of Elements of a Non-empty Sets

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Summary. The first part of the article is a continuation of [2]. Next, we define the identity sequence of natural numbers and the constant sequences. The main part of this article is the definition of tuples. The element of a set of all sequences of the length n of D is called a tuple of a non-empty set D and it is denoted by element of D^n. Also some basic facts about tuples of a non-empty set are proved.

MML Identifier: FINSEQ_2.

The notation and terminology used here have been introduced in the following articles: [9], [8], [6], [1], [10], [4], [5], [2], [3], and [7]. For simplicity we adopt the following rules: i, j, l denote natural numbers, a, b, x_1, x_2, x_3 are arbitrary, D, D', E denote non-empty sets, d, d_1, d_2, d_3 denote elements of D, d', d'_1, d'_2, d'_3 denote elements of D', and p, q, r denote finite sequences. Next we state a number of propositions:

(1) $\min(i,j)$ is a natural number and $\max(i,j)$ is a natural number.

(2) If $l = \min(i,j)$, then $\mathrm{Seg}\, i \cap \mathrm{Seg}\, j = \mathrm{Seg}\, l$.

(3) If $i \leq j$, then $\max(0, i-j) = 0$.

(4) If $j \leq i$, then $\max(0, i-j) = i-j$.

(5) $\max(0, i-j)$ is a natural number.

(6) $\min(0,i) = 0$ and $\min(i,0) = 0$ and $\max(0,i) = i$ and $\max(i,0) = i$.

(7) If $i \neq 0$, then $\mathrm{Seg}\, i$ is a non-empty subset of $\mathbb{N}$.

(8) If $i \in \mathrm{Seg}(l+1)$, then $i \in \mathrm{Seg}\, l$ or $i = l+1$.

(9) If $i \in \mathrm{Seg}\, l$, then $i \in \mathrm{Seg}(l+j)$.

(10) If $\mathrm{len}\, p = i$ and $\mathrm{len}\, q = i$ and for every j such that $j \in \mathrm{Seg}\, i$ holds $p(j) = q(j)$, then $p = q$.

[1]Supported by RPBP.III-24.C1.

ISSN 0777–4028

(11) If $b \in \operatorname{rng} p$, then there exists i such that $i \in \operatorname{Seg}(\operatorname{len} p)$ and $p(i) = b$.

(12) If $i \in \operatorname{Seg}(\operatorname{len} p)$, then $p(i) \in \operatorname{rng} p$.

(13) For every finite sequence p of elements of D such that $i \in \operatorname{Seg}(\operatorname{len} p)$ holds $p(i) \in D$.

(14) If for every i such that $i \in \operatorname{Seg}(\operatorname{len} p)$ holds $p(i) \in D$, then p is a finite sequence of elements of D.

(15) $\langle d_1, d_2 \rangle$ is a finite sequence of elements of D.

(16) $\langle d_1, d_2, d_3 \rangle$ is a finite sequence of elements of D.

(17) If $i \in \operatorname{Seg}(\operatorname{len} p)$, then $(p \frown q)(i) = p(i)$.

(18) If $i \in \operatorname{Seg}(\operatorname{len} p)$, then $i \in \operatorname{Seg}(\operatorname{len}(p \frown q))$.

(19) $\operatorname{len}(p \frown \langle a \rangle) = \operatorname{len} p + 1$.

(20) If $p \frown \langle a \rangle = q \frown \langle b \rangle$, then $p = q$ and $a = b$.

(21) If $\operatorname{len} p = i + 1$, then there exist q, a such that $p = q \frown \langle a \rangle$.

(22) For every finite sequence p of elements of D such that $\operatorname{len} p \neq 0$ there exists a finite sequence q of elements of D and there exists d such that $p = q \frown \langle d \rangle$.

(23) If $q = p \restriction \operatorname{Seg} i$ and $\operatorname{len} p \leq i$, then $p = q$.

(24) If $q = p \restriction \operatorname{Seg} i$, then $\operatorname{len} q = \min(i, \operatorname{len} p)$.

(25) If $\operatorname{len} r = i + j$, then there exist p, q such that $\operatorname{len} p = i$ and $\operatorname{len} q = j$ and $r = p \frown q$.

(26) For every finite sequence r of elements of D such that $\operatorname{len} r = i + j$ there exist finite sequences p, q of elements of D such that $\operatorname{len} p = i$ and $\operatorname{len} q = j$ and $r = p \frown q$.

In the article we present several logical schemes. The scheme *SeqLambdaD* concerns a natural number $\mathcal{A}$, a non-empty set $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{B}$ and states that:

there exists a finite sequence z of elements of $\mathcal{B}$ such that $\operatorname{len} z = \mathcal{A}$ and for every j such that $j \in \operatorname{Seg} \mathcal{A}$ holds $z(j) = \mathcal{F}(j)$

for all values of the parameters.

The scheme *IndSeqD* deals with a non-empty set $\mathcal{A}$, and a unary predicate $\mathcal{P}$, and states that:

for every finite sequence p of elements of $\mathcal{A}$ holds $\mathcal{P}[p]$

provided the parameters meet the following requirements:

- $\mathcal{P}[\varepsilon_{\mathcal{A}}]$,
- for every finite sequence p of elements of $\mathcal{A}$ and for every element x of $\mathcal{A}$ such that $\mathcal{P}[p]$ holds $\mathcal{P}[p \frown \langle x \rangle]$.

We now state a number of propositions:

(27) For every non-empty subset D' of D and for every finite sequence p of elements of D' holds p is a finite sequence of elements of D.

(28) For every function f from $\operatorname{Seg} i$ into D holds f is a finite sequence of elements of D.

(29) p is a function from $\operatorname{Seg}(\operatorname{len} p)$ into $\operatorname{rng} p$.

(30) For every finite sequence p of elements of D holds p is a function from $\mathrm{Seg}(\mathrm{len}\, p)$ into D.

(31) For every function f from $\mathbb{N}$ into D holds $f \restriction \mathrm{Seg}\, i$ is a finite sequence of elements of D.

(32) For every function f from $\mathbb{N}$ into D such that $q = f \restriction \mathrm{Seg}\, i$ holds $\mathrm{len}\, q = i$.

(33) For every function f such that $\mathrm{rng}\, p \subseteq \mathrm{dom}\, f$ and $q = f \cdot p$ holds $\mathrm{len}\, q = \mathrm{len}\, p$.

(34) If $D = \mathrm{Seg}\, i$, then for every finite sequence p and for every finite sequence q of elements of D such that $i \leq \mathrm{len}\, p$ holds $p \cdot q$ is a finite sequence.

(35) If $D = \mathrm{Seg}\, i$, then for every finite sequence p of elements of D' and for every finite sequence q of elements of D such that $i \leq \mathrm{len}\, p$ holds $p \cdot q$ is a finite sequence of elements of D'.

(36) For every finite sequence p of elements of D and for every function f from D into D' holds $f \cdot p$ is a finite sequence of elements of D'.

(37) For every finite sequence p of elements of D and for every function f from D into D' such that $q = f \cdot p$ holds $\mathrm{len}\, q = \mathrm{len}\, p$.

(38) For every function f from D into D' holds $f \cdot \varepsilon_D = \varepsilon_{D'}$.

(39) For every finite sequence p of elements of D and for every function f from D into D' such that $p = \langle x_1 \rangle$ holds $f \cdot p = \langle f(x_1) \rangle$.

(40) For every finite sequence p of elements of D and for every function f from D into D' such that $p = \langle x_1, x_2 \rangle$ holds $f \cdot p = \langle f(x_1), f(x_2) \rangle$.

(41) For every finite sequence p of elements of D and for every function f from D into D' such that $p = \langle x_1, x_2, x_3 \rangle$ holds $f \cdot p = \langle f(x_1), f(x_2), f(x_3) \rangle$.

(42) For every function f from $\mathrm{Seg}\, i$ into $\mathrm{Seg}\, j$ such that if $j = 0$, then $i = 0$ but $j \leq \mathrm{len}\, p$ holds $p \cdot f$ is a finite sequence.

(43) For every function f from $\mathrm{Seg}\, i$ into $\mathrm{Seg}\, i$ such that $i \leq \mathrm{len}\, p$ holds $p \cdot f$ is a finite sequence.

(44) For every function f from $\mathrm{Seg}(\mathrm{len}\, p)$ into $\mathrm{Seg}(\mathrm{len}\, p)$ holds $p \cdot f$ is a finite sequence.

(45) For every function f from $\mathrm{Seg}\, i$ into $\mathrm{Seg}\, i$ such that $\mathrm{rng}\, f = \mathrm{Seg}\, i$ and $i \leq \mathrm{len}\, p$ and $q = p \cdot f$ holds $\mathrm{len}\, q = i$.

(46) For every function f from $\mathrm{Seg}(\mathrm{len}\, p)$ into $\mathrm{Seg}(\mathrm{len}\, p)$ such that $\mathrm{rng}\, f = \mathrm{Seg}(\mathrm{len}\, p)$ and $q = p \cdot f$ holds $\mathrm{len}\, q = \mathrm{len}\, p$.

(47) For every permutation f of $\mathrm{Seg}\, i$ such that $i \leq \mathrm{len}\, p$ and $q = p \cdot f$ holds $\mathrm{len}\, q = i$.

(48) For every permutation f of $\mathrm{Seg}(\mathrm{len}\, p)$ such that $q = p \cdot f$ holds $\mathrm{len}\, q = \mathrm{len}\, p$.

(49) For every finite sequence p of elements of D and for every function f from $\mathrm{Seg}\, i$ into $\mathrm{Seg}\, j$ such that if $j = 0$, then $i = 0$ but $j \leq \mathrm{len}\, p$ holds $p \cdot f$ is a finite sequence of elements of D.

(50) For every finite sequence p of elements of D and for every function f from $\mathrm{Seg}\, i$ into $\mathrm{Seg}\, i$ such that $i \leq \mathrm{len}\, p$ holds $p \cdot f$ is a finite sequence of elements of D.

(51) For every finite sequence p of elements of D and for every function f from $\mathrm{Seg}(\mathrm{len}\, p)$ into $\mathrm{Seg}(\mathrm{len}\, p)$ holds $p \cdot f$ is a finite sequence of elements of D.

(52) $\mathrm{id}_{\mathrm{Seg}\, i}$ is a finite sequence of elements of $\mathbb{N}$.

Let us consider i. The functor id_i yielding a finite sequence, is defined as follows:

$\mathrm{id}_i = \mathrm{id}_{\mathrm{Seg}\, i}$.

One can prove the following propositions:

(53) $\mathrm{id}_i = \mathrm{id}_{\mathrm{Seg}\, i}$.

(54) $\mathrm{dom}(\mathrm{id}_i) = \mathrm{Seg}\, i$.

(55) $\mathrm{len}(\mathrm{id}_i) = i$.

(56) If $j \in \mathrm{Seg}\, i$, then $\mathrm{id}_i(j) = j$.

(57) If $i \neq 0$, then for every element k of $\mathrm{Seg}\, i$ holds $\mathrm{id}_i(k) = k$.

(58) $\mathrm{id}_0 = \varepsilon$.

(59) $\mathrm{id}_1 = \langle 1 \rangle$.

(60) $\mathrm{id}_{i+1} = \mathrm{id}_i \frown \langle i+1 \rangle$.

(61) $\mathrm{id}_2 = \langle 1, 2 \rangle$.

(62) $\mathrm{id}_3 = \langle 1, 2, 3 \rangle$.

(63) $p \cdot \mathrm{id}_i = p \restriction \mathrm{Seg}\, i$.

(64) If $\mathrm{len}\, p \leq i$, then $p \cdot \mathrm{id}_i = p$.

(65) id_i is a permutation of $\mathrm{Seg}\, i$.

(66) $\mathrm{Seg}\, i \longmapsto a$ is a finite sequence.

Let us consider i, a. The functor $i \longmapsto a$ yielding a finite sequence, is defined as follows:

$i \longmapsto a = \mathrm{Seg}\, i \longmapsto a$.

We now state a number of propositions:

(67) $i \longmapsto a = \mathrm{Seg}\, i \longmapsto a$.

(68) $\mathrm{dom}(i \longmapsto a) = \mathrm{Seg}\, i$.

(69) $\mathrm{len}(i \longmapsto a) = i$.

(70) If $j \in \mathrm{Seg}\, i$, then $(i \longmapsto a)(j) = a$.

(71) If $i \neq 0$, then for every element k of $\mathrm{Seg}\, i$ holds $(i \longmapsto d)(k) = d$.

(72) $0 \longmapsto a = \varepsilon$.

(73) $1 \longmapsto a = \langle a \rangle$.

(74) $i + 1 \longmapsto a = (i \longmapsto a) \frown \langle a \rangle$.

(75) $2 \longmapsto a = \langle a, a \rangle$.

(76) $3 \longmapsto a = \langle a, a, a \rangle$.

(77) $i \longmapsto d$ is a finite sequence of elements of D.

(78) For every function F such that $[\!:\operatorname{rng} p, \operatorname{rng} q\,:\!] \subseteq \operatorname{dom} F$ holds $F^\circ(p,q)$ is a finite sequence.

(79) For every function F such that $[\!:\operatorname{rng} p, \operatorname{rng} q\,:\!] \subseteq \operatorname{dom} F$ and $r = F^\circ(p,q)$ holds $\operatorname{len} r = \min(\operatorname{len} p, \operatorname{len} q)$.

(80) For every function F such that $[\!:\{a\}, \operatorname{rng} p\,:\!] \subseteq \operatorname{dom} F$ holds $F^\circ(a,p)$ is a finite sequence.

(81) For every function F such that $[\!:\{a\}, \operatorname{rng} p\,:\!] \subseteq \operatorname{dom} F$ and $r = F^\circ(a,p)$ holds $\operatorname{len} r = \operatorname{len} p$.

(82) For every function F such that $[\!:\operatorname{rng} p, \{a\}\,:\!] \subseteq \operatorname{dom} F$ holds $F^\circ(p,a)$ is a finite sequence.

(83) For every function F such that $[\!:\operatorname{rng} p, \{a\}\,:\!] \subseteq \operatorname{dom} F$ and $r = F^\circ(p,a)$ holds $\operatorname{len} r = \operatorname{len} p$.

(84) For every function F from $[\!:D, D'\,:\!]$ into E and for every finite sequence p of elements of D and for every finite sequence q of elements of D' holds $F^\circ(p,q)$ is a finite sequence of elements of E.

(85) For every function F from $[\!:D, D'\,:\!]$ into E and for every finite sequence p of elements of D and for every finite sequence q of elements of D' such that $r = F^\circ(p,q)$ holds $\operatorname{len} r = \min(\operatorname{len} p, \operatorname{len} q)$.

(86) For every function F from $[\!:D, D'\,:\!]$ into E and for every finite sequence p of elements of D and for every finite sequence q of elements of D' such that $\operatorname{len} p = \operatorname{len} q$ and $r = F^\circ(p,q)$ holds $\operatorname{len} r = \operatorname{len} p$ and $\operatorname{len} r = \operatorname{len} q$.

(87) For every function F from $[\!:D, D'\,:\!]$ into E and for every finite sequence p of elements of D and for every finite sequence p' of elements of D' holds $F^\circ(\varepsilon_D, p') = \varepsilon_E$ and $F^\circ(p, \varepsilon_{D'}) = \varepsilon_E$.

(88) For every function F from $[\!:D, D'\,:\!]$ into E and for every finite sequence p of elements of D and for every finite sequence q of elements of D' such that $p = \langle d_1 \rangle$ and $q = \langle d'_1 \rangle$ holds $F^\circ(p,q) = \langle F(d_1, d'_1) \rangle$.

(89) For every function F from $[\!:D, D'\,:\!]$ into E and for every finite sequence p of elements of D and for every finite sequence q of elements of D' such that $p = \langle d_1, d_2 \rangle$ and $q = \langle d'_1, d'_2 \rangle$ holds $F^\circ(p,q) = \langle F(d_1, d'_1), F(d_2, d'_2) \rangle$.

(90) For every function F from $[\!:D, D'\,:\!]$ into E and for every finite sequence p of elements of D and for every finite sequence q of elements of D' such that $p = \langle d_1, d_2, d_3 \rangle$ and $q = \langle d'_1, d'_2, d'_3 \rangle$ holds $F^\circ(p,q) = \langle F(d_1, d'_1), F(d_2, d'_2), F(d_3, d'_3) \rangle$.

(91) For every function F from $[\!:D, D'\,:\!]$ into E and for every finite sequence p of elements of D' holds $F^\circ(d,p)$ is a finite sequence of elements of E.

(92) For every function F from $[\!:D, D'\,:\!]$ into E and for every finite sequence p of elements of D' such that $r = F^\circ(d,p)$ holds $\operatorname{len} r = \operatorname{len} p$.

(93) For every function F from $[\!:D, D'\,:\!]$ into E holds $F^\circ(d, \varepsilon_{D'}) = \varepsilon_E$.

(94) For every function F from $[\!:D, D'\,:\!]$ into E and for every finite sequence p of elements of D' such that $p = \langle d'_1 \rangle$ holds $F^\circ(d,p) = \langle F(d, d'_1) \rangle$.

(95) For every function F from $[\!:D, D'\,:\!]$ into E and for every finite sequence

p of elements of D' such that $p = \langle d'_1, d'_2 \rangle$ holds $F^\circ(d, p) = \langle F(d, d'_1), F(d, d'_2) \rangle$.

(96) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D' such that $p = \langle d'_1, d'_2, d'_3 \rangle$ holds $F^\circ(d, p) = \langle F(d, d'_1), F(d, d'_2), F(d, d'_3) \rangle$.

(97) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D holds $F^\circ(p, d')$ is a finite sequence of elements of E.

(98) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D such that $r = F^\circ(p, d')$ holds $\operatorname{len} r = \operatorname{len} p$.

(99) For every function F from $[: D, D' :]$ into E holds $F^\circ(\varepsilon_D, d') = \varepsilon_E$.

(100) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D such that $p = \langle d_1 \rangle$ holds $F^\circ(p, d') = \langle F(d_1, d') \rangle$.

(101) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D such that $p = \langle d_1, d_2 \rangle$ holds $F^\circ(p, d') = \langle F(d_1, d'), F(d_2, d') \rangle$.

(102) For every function F from $[: D, D' :]$ into E and for every finite sequence p of elements of D such that $p = \langle d_1, d_2, d_3 \rangle$ holds $F^\circ(p, d') = \langle F(d_1, d'), F(d_2, d'), F(d_3, d') \rangle$.

Let us consider D. A non-empty set is said to be a non-empty set of finite sequences of D if:

if $a \in$ it, then a is a finite sequence of elements of D.

We now state two propositions:

(103) For all D, D' holds D' is a non-empty set of finite sequences of D if and only if for every a such that $a \in D'$ holds a is a finite sequence of elements of D.

(104) D^* is a non-empty set of finite sequences of D.

Let us consider D. Then D^* is a non-empty set of finite sequences of D.

Next we state two propositions:

(105) For every non-empty set D' of finite sequences of D holds $D' \subseteq D^*$.

(106) For every non-empty set S of finite sequences of D and for every element s of S holds s is a finite sequence of elements of D.

Let us consider D, and let S be a non-empty set of finite sequences of D. We see that it makes sense to consider the following mode for restricted scopes of arguments. Then all the objests of the mode element of S are a finite sequence of elements of D.

One can prove the following proposition

(107) For every non-empty subset D' of D and for every non-empty set S of finite sequences of D' holds S is a non-empty set of finite sequences of D.

In the sequel s is an element of D^*. Let us consider i, D. The functor D^i yielding a non-empty set of finite sequences of D, is defined as follows:

$D^i = \{s : \operatorname{len} s = i\}$.

Next we state a number of propositions:

(108) $D^i = \{s : \operatorname{len} s = i\}$.

(109) For every element z of D^i holds $\operatorname{len} z = i$.

(110) For every finite sequence z of elements of D holds z is an element of $D^{\operatorname{len} z}$.

(111) $D^i = D^{\operatorname{Seg} i}$.

(112) $D^0 = \{\varepsilon_D\}$.

(113) For every element z of D^0 holds $z = \varepsilon_D$.

(114) ε_D is an element of D^0.

(115) For every element z of D^0 and for every element t of D^i holds $z \frown t = t$ and $t \frown z = t$.

(116) $D^1 = \{\langle d \rangle\}$.

(117) For every element z of D^1 there exists d such that $z = \langle d \rangle$.

(118) $\langle d \rangle$ is an element of D^1.

(119) $D^2 = \{\langle d_1, d_2 \rangle\}$.

(120) For every element z of D^2 there exist d_1, d_2 such that $z = \langle d_1, d_2 \rangle$.

(121) $\langle d_1, d_2 \rangle$ is an element of D^2.

(122) $D^3 = \{\langle d_1, d_2, d_3 \rangle\}$.

(123) For every element z of D^3 there exist d_1, d_2, d_3 such that $z = \langle d_1, d_2, d_3 \rangle$.

(124) $\langle d_1, d_2, d_3 \rangle$ is an element of D^3.

(125) $D^{i+j} = \{z \frown t\}$.

(126) For every element s of D^{i+j} there exists an element z of D^i and there exists an element t of D^j such that $s = z \frown t$.

(127) For every element z of D^i and for every element t of D^j holds $z \frown t$ is an element of D^{i+j}.

(128) $D^* = \bigcup\{D^i\}$.

(129) For every non-empty subset D' of D and for every element z of D'^i holds z is an element of D^i.

(130) If $D^i = D^j$, then $i = j$.

(131) id_i is an element of $\mathbb{N}^i$.

(132) $i \longmapsto d$ is an element of D^i.

(133) For every element z of D^i and for every function f from D into D' holds $f \cdot z$ is an element of D'^i.

(134) For every element z of D^i and for every function f from $\operatorname{Seg} i$ into $\operatorname{Seg} i$ such that $\operatorname{rng} f = \operatorname{Seg} i$ holds $z \cdot f$ is an element of D^i.

(135) For every element z of D^i and for every permutation f of $\operatorname{Seg} i$ holds $z \cdot f$ is an element of D^i.

(136) For every element z of D^i and for every d holds $(z \frown \langle d \rangle)(i+1) = d$.

(137) For every element z of D^{i+1} there exists an element t of D^i and there exists d such that $z = t \frown \langle d \rangle$.

(138) For every element z of D^i holds $z \cdot \mathrm{id}_i = z$.

(139) For all elements z_1, z_2 of D^i such that for every j such that $j \in \mathrm{Seg}\, i$ holds $z_1(j) = z_2(j)$ holds $z_1 = z_2$.

(140) For every function F from $[: D, D' :]$ into E and for every element z_1 of D^i and for every element z_2 of D'^i holds $F^\circ(z_1, z_2)$ is an element of E^i.

(141) For every function F from $[: D, D' :]$ into E and for every element z of D'^i holds $F^\circ(d, z)$ is an element of E^i.

(142) For every function F from $[: D, D' :]$ into E and for every element z of D^i holds $F^\circ(z, d')$ is an element of E^i.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(**1**):41–46, 1990.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(**1**):107–114, 1990.

[3] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(**1**):175–180, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(**1**):55–65, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(**1**):153–164, 1990.

[6] Krzysztof Hryniewiecki. Basic properties of real numbers. *Formalized Mathematics*, 1(**1**):35–40, 1990.

[7] Andrzej Trybulec. Binary operations applied to functions. *Formalized Mathematics*, 1(**2**):329–334, 1990.

[8] Andrzej Trybulec. Domains and their Cartesian products. *Formalized Mathematics*, 1(**1**):115–122, 1990.

[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1(**1**):9–11, 1990.

[10] Andrzej Trybulec and Czesław Byliński. Some properties of real numbers. *Formalized Mathematics*, 1(**3**):445–449, 1990.

Received March 1, 1990