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Equivalence Relations and Classes of Abstraction [1]

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Summary. In this article we deal with the notion of equivalence relation. The main properties of equivalence relations are proved. Then we define the classes of abstraction determined by an equivalence relation. Finally, the connections between a partition of a set and an equivalence relation are presented. We introduce the following notation of modes: *Equivalence Relation, a partition*.

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The notation and terminology used in this paper are introduced in the following articles: [6], [7], [9], [8], [5], [3], [2], [4], and [1]. For simplicity we adopt the following rules: x, y, z are arbitrary, i, j are natural numbers, X, Y are sets, A, B are subsets of X, R, R_1, R_2 are relations on X, and $SFXX$ is a family of subsets of $[:X, X:]$. The following two propositions are true:

(1) If $i < j$, then $j - i$ is a natural number.

(2) For every Y such that $Y \subseteq [:X, X:]$ holds Y is a relation on X.

Let us consider X. The functor ∇_X yielding a relation on X, is defined as follows:

$\nabla_X = [:X, X:]$.

We now state a proposition

(3) $\nabla_X = [:X, X:]$.

Let us consider X, R_1, R_2. Then $R_1 \cap R_2$ is a relation on X. Then $R_1 \cup R_2$ is a relation on X.

Next we state a proposition

(4) $\triangle_X$ is reflexive in X and $\triangle_X$ is symmetric in X and $\triangle_X$ is transitive in X.

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Let us consider X. A relation on X is called an equivalence relation of X if: it is reflexive in X and it is symmetric in X and it is transitive in X.

The following three propositions are true:

(5) R is an equivalence relation of X if and only if R is reflexive in X and R is symmetric in X and R is transitive in X.

(6) $\triangle_X$ is an equivalence relation of X.

(7) ∇_X is an equivalence relation of X.

Let us consider X. Then $\triangle_X$ is an equivalence relation of X. Then ∇_X is an equivalence relation of X.

In the sequel EqR, EqR_1, EqR_2 will be equivalence relations of X. We now state several propositions:

(8) EqR is reflexive in X.

(9) EqR is symmetric in X.

(10) EqR is transitive in X.

(11) If $x \in X$, then $\langle x, x\rangle \in EqR$.

(12) If $\langle x, y\rangle \in EqR$, then $\langle y, x\rangle \in EqR$.

(13) If $\langle x, y\rangle \in EqR$ and $\langle y, z\rangle \in EqR$, then $\langle x, z\rangle \in EqR$.

(14) If there exists x such that $x \in X$, then $EqR \neq \varnothing$.

(15) $\mathrm{field}\, EqR = X$.

(16) R is an equivalence relation of X if and only if R is pseudo reflexive and R is symmetric and R is transitive and $\mathrm{field}\, R = X$.

Let us consider X, EqR_1, EqR_2. Then $EqR_1 \cap EqR_2$ is an equivalence relation of X.

We now state four propositions:

(17) $\triangle_X \cap EqR = \triangle_X$.

(18) $(\nabla_X) \cap R = R$.

(19) For every $SFXX$ such that $SFXX \neq \emptyset$ and for every Y such that $Y \in SFXX$ holds Y is an equivalence relation of X holds $\bigcap SFXX$ is an equivalence relation of X.

(20) For every R there exists EqR such that $R \subseteq EqR$ and for every EqR_2 such that $R \subseteq EqR_2$ holds $EqR \subseteq EqR_2$.

Let us consider X, EqR_1, EqR_2. The functor $EqR_1 \sqcup EqR_2$ yielding an equivalence relation of X, is defined by:

$EqR_1 \cup EqR_2 \subseteq EqR_1 \sqcup EqR_2$ and for every EqR such that $EqR_1 \cup EqR_2 \subseteq EqR$ holds $EqR_1 \sqcup EqR_2 \subseteq EqR$.

Next we state several propositions:

(21) For every equivalence relation R of X holds $R = EqR_1 \sqcup EqR_2$ if and only if $EqR_1 \cup EqR_2 \subseteq R$ and for every EqR such that $EqR_1 \cup EqR_2 \subseteq EqR$ holds $R \subseteq EqR$.

(22) $EqR \sqcup EqR = EqR$.

(23) $EqR_1 \sqcup EqR_2 = EqR_2 \sqcup EqR_1$.

(24) $EqR_1 \cap (EqR_1 \sqcup EqR_2) = EqR_1$.

(25) $EqR_1 \sqcup (EqR_1 \cap EqR_2) = EqR_1$.

The scheme *Ex_Eq_Rel* concerns a set $\mathcal{A}$, and a binary predicate $\mathcal{P}$, and states that:

there exists an equivalence relation EqR of $\mathcal{A}$ such that for all x, y holds $\langle x, y \rangle \in EqR$ if and only if $x \in \mathcal{A}$ and $y \in \mathcal{A}$ and $\mathcal{P}[x, y]$

provided the parameters satisfy the following conditions:

- for every x such that $x \in \mathcal{A}$ holds $\mathcal{P}[x, x]$,
- for all x, y such that $\mathcal{P}[x, y]$ holds $\mathcal{P}[y, x]$,
- for all x, y, z such that $\mathcal{P}[x, y]$ and $\mathcal{P}[y, z]$ holds $\mathcal{P}[x, z]$.

Let us consider X, EqR, x. The functor $[x]_{EqR}$ yielding a subset of X, is defined by:

$[x]_{EqR} = EqR^\circ \{x\}$.

We now state a number of propositions:

(26) $[x]_{EqR} = EqR^\circ \{x\}$.

(27) $y \in [x]_{EqR}$ if and only if $\langle y, x \rangle \in EqR$.

(28) For every x such that $x \in X$ holds $x \in [x]_{EqR}$.

(29) For every x such that $x \in X$ there exists y such that $x \in [y]_{EqR}$.

(30) If $y \in [x]_{EqR}$ and $z \in [x]_{EqR}$, then $\langle y, z \rangle \in EqR$.

(31) For every x such that $x \in X$ holds $y \in [x]_{EqR}$ if and only if $[x]_{EqR} = [y]_{EqR}$.

(32) For all x, y such that $x \in X$ and $y \in X$ holds $[x]_{EqR} = [y]_{EqR}$ or $[x]_{EqR}$ misses $[y]_{EqR}$.

(33) For every x such that $x \in X$ holds $[x]_{\triangle_X} = \{x\}$.

(34) For every x such that $x \in X$ holds $[x]_{\nabla_X} = X$.

(35) If there exists x such that $[x]_{EqR} = X$, then $EqR = \nabla_X$.

(36) Suppose $x \in X$. Then $\langle x, y \rangle \in EqR_1 \sqcup EqR_2$ if and only if there exists a finite sequence f such that $1 \leq \operatorname{len} f$ and $x = f(1)$ and $y = f(\operatorname{len} f)$ and for every i such that $1 \leq i$ and $i < \operatorname{len} f$ holds $\langle f(i), f(i+1) \rangle \in EqR_1 \cup EqR_2$.

(37) For every equivalence relation E of X such that $E = EqR_1 \cup EqR_2$ for every x such that $x \in X$ holds $[x]_E = [x]_{EqR_1}$ or $[x]_E = [x]_{EqR_2}$.

(38) If $EqR_1 \cup EqR_2 = \nabla_X$, then $EqR_1 = \nabla_X$ or $EqR_2 = \nabla_X$.

Let us consider X, EqR. The functor $\operatorname{Classes} EqR$ yields a family of subsets of X and is defined as follows:

$A \in \operatorname{Classes} EqR$ if and only if there exists x such that $x \in X$ and $A = [x]_{EqR}$.

The following two propositions are true:

(39) $A \in \operatorname{Classes} EqR$ if and only if there exists x such that $x \in X$ and $A = [x]_{EqR}$.

(40) If $X = \emptyset$, then $\operatorname{Classes} EqR = \emptyset$.

Let us consider X. A family of subsets of X is said to be a partition of X if:

$\bigcup$it $= X$ and for every A such that $A \in$ it holds $A \neq \emptyset$ and for every B such that $B \in$ it holds $A = B$ or A misses B if $X \neq \emptyset$, it $= \emptyset$, otherwise.

We now state several propositions:

(41) If $X \neq \emptyset$, then for every family F of subsets of X holds F is a partition of X if and only if $\bigcup F = X$ and for every A such that $A \in F$ holds $A \neq \emptyset$ and for every B such that $B \in F$ holds $A = B$ or A misses B.

(42) Classes EqR is a partition of X.

(43) For every partition P of X there exists EqR such that $P =$ Classes EqR.

(44) For every x such that $x \in X$ holds $\langle x, y \rangle \in EqR$ if and only if $[x]_{EqR} = [y]_{EqR}$.

(45) If $x \in$ Classes EqR, then there exists an element y of X such that $x = [y]_{EqR}$.

(46) For every x such that $x \in X$ holds $[x]_{EqR} \in$ Classes EqR.

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