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Properties of ZF Models

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Summary. The article deals with the concepts of satisfiability of ZF set theory language formulae in a model (a non-empty family of sets) and the axioms of ZF theory introduced in [6]. It is shown that the transitive model satisfies the axiom of extensionality and that it satisfies the axiom of pairs if and only if it is closed to pair operation; it satisfies the axiom of unions if and only if it is closed to union operation, ect. The conditions which are satisfied by arbitrary model of ZF set theory are also shown. Besides introduced are definable and parametrically definable functions.

MML Identifier: `ZFMODEL1`.

The notation and terminology used in this paper are introduced in the following papers: [8], [4], [1], [5], [7], [3], and [2]. For simplicity we follow a convention: x, y, z will be variables, H will be a ZF-formula, E will be a non-empty family of sets, X, Y, Z will be sets, u, v, w will be elements of E, and f, g will be functions from VAR into E. One can prove the following propositions:

(1) If E is transitive, then $E \models$ the axiom of extensionality.

(2) If E is transitive, then $E \models$ the axiom of pairs if and only if for all u, v holds $\{u, v\} \in E$.

(3) If E is transitive, then $E \models$ the axiom of pairs if and only if for all X, Y such that $X \in E$ and $Y \in E$ holds $\{X, Y\} \in E$.

(4) If E is transitive, then $E \models$ the axiom of unions if and only if for every u holds $\bigcup u \in E$.

(5) If E is transitive, then $E \models$ the axiom of unions if and only if for every X such that $X \in E$ holds $\bigcup X \in E$.

(6) If E is transitive, then $E \models$ the axiom of infinity if and only if there exists u such that $u \neq \emptyset$ and for every v such that $v \in u$ there exists w such that $v \subseteq w$ and $v \neq w$ and $w \in u$.

(7) If E is transitive, then $E \models$ the axiom of infinity if and only if there exists X such that $X \in E$ and $X \neq \emptyset$ and for every Y such that $Y \in X$ there exists Z such that $Y \subseteq Z$ and $Y \neq Z$ and $Z \in X$.

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(8) If E is transitive, then $E \models$ the axiom of power sets if and only if for every u holds $E \cap 2^u \in E$.

(9) If E is transitive, then $E \models$ the axiom of power sets if and only if for every X such that $X \in E$ holds $E \cap 2^X \in E$.

(10) If $x \notin \mathrm{Free}\, H$ and $E, f \models H$, then $E, f \models \forall_x H$.

(11) If $\{x, y\}$ misses $\mathrm{Free}\, H$ and $E, f \models H$, then $E, f \models \forall_{x,y} H$.

(12) If $\{x, y, z\}$ misses $\mathrm{Free}\, H$ and $E, f \models H$, then $E, f \models \forall_{x,y,z} H$.

The arguments of the notions defined below are the following: H, E which are objects of the type reserved above; val which is a function from VAR into E. Let us assume that $x_0 \notin \mathrm{Free}\, H$ and $E, val \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$. The functor $\mathrm{f}_H[val]$ yielding a function from E into E, is defined by:

for every g such that for every y such that $g(y) \neq val(y)$ holds $x_0 = y$ or $x_3 = y$ or $x_4 = y$ holds $E, g \models H$ if and only if $\mathrm{f}_H[val](g(x_3)) = g(x_4)$.

Next we state two propositions:

(13) Suppose $x_0 \notin \mathrm{Free}\, H$ and $E, f \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$. Let F be a function from E into E. Then $F = \mathrm{f}_H[f]$ if and only if for every g such that for every y such that $g(y) \neq f(y)$ holds $x_0 = y$ or $x_3 = y$ or $x_4 = y$ holds $E, g \models H$ if and only if $F(g(x_3)) = g(x_4)$.

(14) For all H, f, g such that for every x such that $f(x) \neq g(x)$ holds $x \notin \mathrm{Free}\, H$ and $E, f \models H$ holds $E, g \models H$.

Let us consider H, E. Let us assume that $\mathrm{Free}\, H \subseteq \{x_3, x_4\}$ and $E \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$. The functor $\mathrm{f}_H[E]$ yielding a function from E into E, is defined by:

for every g holds $E, g \models H$ if and only if $\mathrm{f}_H[E](g(x_3)) = g(x_4)$.

The following proposition is true

(15) Suppose $\mathrm{Free}\, H \subseteq \{x_3, x_4\}$ and $E \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$. Then for every function F from E into E holds $F = \mathrm{f}_H[E]$ if and only if for every g holds $E, g \models H$ if and only if $F(g(x_3)) = g(x_4)$.

We now define two new predicates. The arguments of the notions defined below are the following: F which is a function; E which is an object of the type reserved above. The predicate F is definable in E is defined by:

there exists H such that $\mathrm{Free}\, H \subseteq \{x_3, x_4\}$ and $E \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$ and $F = \mathrm{f}_H[E]$.

The predicate F is parametrically definable in E is defined by:

there exist H, f such that $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\, H$ and
$E, f \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$
and $F = \mathrm{f}_H[f]$.

One can prove the following propositions:

(16) For every function F holds F is definable in E if and only if there exists H such that $\mathrm{Free}\, H \subseteq \{x_3, x_4\}$ and $E \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$ and $F = \mathrm{f}_H[E]$.

(17) For every function F holds F is parametrically definable in E if and only if there exist H, f such that $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\, H$ and $E, f \models$

$\forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$ and $F = \mathrm{f}_H[f]$.

(18) For every function F such that F is definable in E holds F is parametrically definable in E.

(19) Suppose E is transitive. Then for every H such that $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\, H$ holds $E \models$ the axiom of substitution for H if and only if for all H, f such that $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\, H$ and $E, f \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$ for every u holds $\mathrm{f}_H[f] \circ u \in E$.

(20) If E is transitive, then for every H such that $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\, H$ holds $E \models$ the axiom of substitution for H if and only if for every function F such that F is parametrically definable in E for every X such that $X \in E$ holds $F \circ X \in E$.

(21) Suppose E is a model of ZF. Then

(i) E is transitive,

(ii) for all u, v such that for every w holds $w \in u$ if and only if $w \in v$ holds $u = v$,

(iii) for all u, v holds $\{u, v\} \in E$,

(iv) for every u holds $\bigcup u \in E$,

(v) there exists u such that $u \neq \emptyset$ and for every v such that $v \in u$ there exists w such that $v \subseteq w$ and $v \neq w$ and $w \in u$,

(vi) for every u holds $E \cap 2^u \in E$,

(vii) for all H, f such that $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\, H$ and
$E, f \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$
for every u holds $\mathrm{f}_H[f] \circ u \in E$.

(22) Suppose that

(i) E is transitive,

(ii) for all u, v holds $\{u, v\} \in E$,

(iii) for every u holds $\bigcup u \in E$,

(iv) there exists u such that $u \neq \emptyset$ and for every v such that $v \in u$ there exists w such that $v \subseteq w$ and $v \neq w$ and $w \in u$,

(v) for every u holds $E \cap 2^u \in E$,

(vi) for all H, f such that $\{x_0, x_1, x_2\}$ misses $\mathrm{Free}\, H$ and
$E, f \models \forall_{x_3}(\exists_{x_0}(\forall_{x_4} H \Leftrightarrow x_4{=}x_0))$
for every u holds $\mathrm{f}_H[f] \circ u \in E$.

Then E is a model of ZF.

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