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Binary Operations Applied to Functions

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Summary. In the article we introduce functors yielding to a binary operation its composition with an arbitrary functions on its left side, its right side or both. We prove theorems describing the basic properties of these functors. We introduce also constant functions and converse of a function. The recent concept is defined for an arbitrary function, however is meaningful in the case of functions which range is a subset of a Cartesian product of two sets. Then the converse of a function has the same domain as the function itself and assigns to an element of the domain the mirror image of the ordered pair assigned by the function. In the case of functions defined on a non-empty set we redefine the above mentioned functors and prove simplified versions of theorems proved in the general case. We prove also theorems stating relationships between introduced concepts and such properties of binary operations as commutativity or associativity.

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The notation and terminology used in this paper have been introduced in the following articles: [6], [7], [3], [4], [1], [8], [2], [5], and [9]. One can prove the following proposition

(1) For every relation R for all sets A, B such that $A \neq \emptyset$ and $B \neq \emptyset$ and $R = [: A, B :]$ holds $\operatorname{dom} R = A$ and $\operatorname{rng} R = B$.

In the sequel f, g, h will be functions and A will be a set. Next we state three propositions:

(2) $\delta_A = \langle \mathrm{id}_A, \mathrm{id}_A \rangle$.

(3) If $\operatorname{dom} f = \operatorname{dom} g$, then $\operatorname{dom}(f \cdot h) = \operatorname{dom}(g \cdot h)$.

(4) If $\operatorname{dom} f = \emptyset$ and $\operatorname{dom} g = \emptyset$, then $f = g$.

We adopt the following convention: F, f, g, h denote functions, A, B denote sets, and x, y, z are arbitrary. Let us consider f. The functor $f^{\smile}$ yields a function and is defined by:

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(i) $\operatorname{dom}(f^{\smile}) = \operatorname{dom} f$,
(ii) for every x such that $x \in \operatorname{dom} f$ holds for all y, z such that $f(x) = \langle y, z \rangle$ holds $(f^{\smile})(x) = \langle z, y \rangle$ but $f(x) = (f^{\smile})(x)$ or there exist y, z such that $f(x) = \langle y, z \rangle$.

We now state several propositions:

(5) Given f, g. Then $g = f^{\smile}$ if and only if the following conditions are satisfied:
(i) $\operatorname{dom} g = \operatorname{dom} f$,
(ii) for every x such that $x \in \operatorname{dom} f$ holds for all y, z such that $f(x) = \langle y, z \rangle$ holds $g(x) = \langle z, y \rangle$ but $f(x) = g(x)$ or there exist y, z such that $f(x) = \langle y, z \rangle$.

(6) $\langle f, g \rangle = \langle g, f \rangle^{\smile}$.

(7) $(f \upharpoonright A)^{\smile} = f^{\smile} \upharpoonright A$.

(8) $(f^{\smile})^{\smile} = f$.

(9) $(\delta_A)^{\smile} = \delta_A$.

(10) $\langle f, g \rangle \upharpoonright A = \langle f \upharpoonright A, g \rangle$.

(11) $\langle f, g \rangle \upharpoonright A = \langle f, g \upharpoonright A \rangle$.

The arguments of the notions defined below are the following: A which is a set; z which is any. The functor $A \longmapsto z$ yields a function and is defined by:

$\operatorname{graph}(A \longmapsto z) = [\!: A, \{z\} :\!]$.

The following propositions are true:

(12) $f = A \longmapsto x$ if and only if $\operatorname{graph} f = [\!: A, \{x\} :\!]$.

(13) If $x \in A$, then $(A \longmapsto z)(x) = z$.

(14) If $A \neq \emptyset$ and $f = A \longmapsto x$, then $\operatorname{dom} f = A$ and $\operatorname{rng} f = \{x\}$.

(15) If $\operatorname{dom} f = A$ and $\operatorname{rng} f = \{x\}$, then $f = A \longmapsto x$.

(16) $\operatorname{dom}(\emptyset \longmapsto x) = \emptyset$ and $\operatorname{rng}(\emptyset \longmapsto x) = \emptyset$.

(17) If for every z such that $z \in \operatorname{dom} f$ holds $f(z) = x$, then $f = \operatorname{dom} f \longmapsto x$.

(18) $(A \longmapsto x) \upharpoonright B = A \cap B \longmapsto x$.

(19) $\operatorname{dom}(A \longmapsto x) = A$ and $\operatorname{rng}(A \longmapsto x) \subseteq \{x\}$.

(20) If $x \in B$, then $(A \longmapsto x)^{-1} B = A$.

(21) $(A \longmapsto x)^{-1} \{x\} = A$.

(22) If $x \notin B$, then $(A \longmapsto x)^{-1} B = \emptyset$.

(23) If $x \in \operatorname{dom} h$, then $h \cdot (A \longmapsto x) = A \longmapsto h(x)$.

(24) If $A \neq \emptyset$ and $x \in \operatorname{dom} h$, then $\operatorname{dom}(h \cdot (A \longmapsto x)) \neq \emptyset$.

(25) $(A \longmapsto x) \cdot h = h^{-1} A \longmapsto x$.

(26) $(A \longmapsto \langle x, y \rangle)^{\smile} = A \longmapsto \langle y, x \rangle$.

Let us consider F, f, g. The functor $F^{\circ}(f, g)$ yields a function and is defined by:

$F^{\circ}(f, g) = F \cdot \langle f, g \rangle$.

The following propositions are true:

(27) $F^{\circ}(f, g) = F \cdot \langle f, g \rangle$.

(28) If $x \in \operatorname{dom}(F^\circ(f,g))$, then $(F^\circ(f,g))(x) = F(f(x), g(x))$.

(29) If $f \restriction A = g \restriction A$, then $(F^\circ(f,h)) \restriction A = (F^\circ(g,h)) \restriction A$.

(30) If $f \restriction A = g \restriction A$, then $(F^\circ(h,f)) \restriction A = (F^\circ(h,g)) \restriction A$.

(31) $F^\circ(f,g) \cdot h = F^\circ(f \cdot h, g \cdot h)$.

(32) $h \cdot F^\circ(f,g) = (h \cdot F)^\circ(f,g)$.

Let us consider F, f, x. The functor $F^\circ(f,x)$ yielding a function, is defined by:

$F^\circ(f,x) = F \cdot \langle f, \operatorname{dom} f \longmapsto x \rangle$.

Next we state several propositions:

(33) $F^\circ(f,x) = F \cdot \langle f, \operatorname{dom} f \longmapsto x \rangle$.

(34) $F^\circ(f,x) = F^\circ(f, \operatorname{dom} f \longmapsto x)$.

(35) If $x \in \operatorname{dom}(F^\circ(f,z))$, then $(F^\circ(f,z))(x) = F(f(x), z)$.

(36) If $f \restriction A = g \restriction A$, then $(F^\circ(f,x)) \restriction A = (F^\circ(g,x)) \restriction A$.

(37) $F^\circ(f,x) \cdot h = F^\circ(f \cdot h, x)$.

(38) $h \cdot F^\circ(f,x) = (h \cdot F)^\circ(f,x)$.

(39) $F^\circ(f,x) \cdot \mathrm{id}_A = F^\circ(f \restriction A, x)$.

Let us consider F, x, g. The functor $F^\circ(x,g)$ yields a function and is defined by:

$F^\circ(x,g) = F \cdot \langle \operatorname{dom} g \longmapsto x, g \rangle$.

We now state several propositions:

(40) $F^\circ(x,g) = F \cdot \langle \operatorname{dom} g \longmapsto x, g \rangle$.

(41) $F^\circ(x,g) = F^\circ(\operatorname{dom} g \longmapsto x, g)$.

(42) If $x \in \operatorname{dom}(F^\circ(z,f))$, then $(F^\circ(z,f))(x) = F(z, f(x))$.

(43) If $f \restriction A = g \restriction A$, then $(F^\circ(x,f)) \restriction A = (F^\circ(x,g)) \restriction A$.

(44) $F^\circ(x,f) \cdot h = F^\circ(x, f \cdot h)$.

(45) $h \cdot F^\circ(x,f) = (h \cdot F)^\circ(x,f)$.

(46) $F^\circ(x,f) \cdot \mathrm{id}_A = F^\circ(x, f \restriction A)$.

For simplicity we follow a convention: X, Y, Z will denote non-empty sets, F will denote a binary operation on X, f, g, h will denote functions from Y into X, and x, x_1, x_2 will denote elements of X. Let us consider X. Then id_X is a function from X into X.

We now state a proposition

(47) $F^\circ(f,g)$ is a function from Y into X.

The arguments of the notions defined below are the following: X, Z which are non-empty sets; F which is a binary operation on X; f, g which are functions from Z into X. Then $F^\circ(f,g)$ is a function from Z into X.

We now state a number of propositions:

(48) For every element z of Y holds $(F^\circ(f,g))(z) = F(f(z), g(z))$.

(49) For every function h from Y into X such that for every element z of Y holds $h(z) = F(f(z), g(z))$ holds $h = F^\circ(f,g)$.

(50) For every function h from Z into Y holds $F^\circ(f,g) \cdot h = F^\circ(f \cdot h, g \cdot h)$.

(51) For every function g from X into X holds $F^\circ(\mathrm{id}_X, g) \cdot f = F^\circ(f, g \cdot f)$.

(52) For every function g from X into X holds $F^\circ(g, \mathrm{id}_X) \cdot f = F^\circ(g \cdot f, f)$.

(53) $F^\circ(\mathrm{id}_X, \mathrm{id}_X) \cdot f = F^\circ(f, f)$.

(54) For every function g from X into X holds $(F^\circ(\mathrm{id}_X, g))(x) = F(x, g(x))$.

(55) For every function g from X into X holds $(F^\circ(g, \mathrm{id}_X))(x) = F(g(x), x)$.

(56) $(F^\circ(\mathrm{id}_X, \mathrm{id}_X))(x) = F(x, x)$.

(57) For all A, B for arbitrary x such that $x \in B$ holds $A \longmapsto x$ is a function from A into B.

(58) For all A, X, x holds $A \longmapsto x$ is a function from A into X.

(59) $F^\circ(f, x)$ is a function from Y into X.

The arguments of the notions defined below are the following: X, Z which are non-empty sets; F which is a binary operation on X; f which is a function from Z into X; x which is an element of X. Then $F^\circ(f, x)$ is a function from Z into X.

The following propositions are true:

(60) For every element y of Y holds $(F^\circ(f, x))(y) = F(f(y), x)$.

(61) If for every element y of Y holds $g(y) = F(f(y), x)$, then $g = F^\circ(f, x)$.

(62) For every function g from Z into Y holds $F^\circ(f, x) \cdot g = F^\circ(f \cdot g, x)$.

(63) $F^\circ(\mathrm{id}_X, x) \cdot f = F^\circ(f, x)$.

(64) $(F^\circ(\mathrm{id}_X, x))(x) = F(x, x)$.

(65) $F^\circ(x, g)$ is a function from Y into X.

The arguments of the notions defined below are the following: X, Z which are non-empty sets; F which is a binary operation on X; x which is an element of X; g which is a function from Z into X. Then $F^\circ(x, g)$ is a function from Z into X.

The following propositions are true:

(66) For every element y of Y holds $(F^\circ(x, f))(y) = F(x, f(y))$.

(67) If for every element y of Y holds $g(y) = F(x, f(y))$, then $g = F^\circ(x, f)$.

(68) For every function g from Z into Y holds $F^\circ(x, f) \cdot g = F^\circ(x, f \cdot g)$.

(69) $F^\circ(x, \mathrm{id}_X) \cdot f = F^\circ(x, f)$.

(70) $(F^\circ(x, \mathrm{id}_X))(x) = F(x, x)$.

(71) For all non-empty sets X, Y, Z for every function f from X into $[\!: Y, Z :\!]$ for every element x of X holds $f^\smile(x) = \langle (f(x))_{\mathbf{2}}, (f(x))_{\mathbf{1}} \rangle$.

(72) For all non-empty sets X, Y, Z for every function f from X into $[\!: Y, Z :\!]$ holds $\mathrm{rng}\, f$ is a relation between Y and Z.

The arguments of the notions defined below are the following: X, Y, Z which are non-empty sets; f which is a function from X into $[\!: Y, Z :\!]$. Then $\mathrm{rng}\, f$ is a relation between Y and Z.

The arguments of the notions defined below are the following: X, Y, Z which are non-empty sets; f which is a function from X into $[\!: Y, Z :\!]$. Then $f^\smile$ is a function from X into $[\!: Z, Y :\!]$.

We now state a proposition

(73) For all non-empty sets X, Y, Z for every function f from X into $[\!: Y, Z :\!]$ holds $\operatorname{rng}(f^{\smile}) = (\operatorname{rng} f)^{\smile}$.

In the sequel y denotes an element of Y. One can prove the following propositions:

(74) If F is associative, then $F^{\circ}(F^{\circ}(x_1, f), x_2) = F^{\circ}(x_1, F^{\circ}(f, x_2))$.

(75) If F is associative, then $F^{\circ}(F^{\circ}(f, x), g) = F^{\circ}(f, F^{\circ}(x, g))$.

(76) If F is associative, then $F^{\circ}(F^{\circ}(f, g), h) = F^{\circ}(f, F^{\circ}(g, h))$.

(77) If F is associative, then $F^{\circ}(F(x_1, x_2), f) = F^{\circ}(x_1, F^{\circ}(x_2, f))$.

(78) If F is associative, then $F^{\circ}(f, F(x_1, x_2)) = F^{\circ}(F^{\circ}(f, x_1), x_2)$.

(79) If F is commutative, then $F^{\circ}(x, f) = F^{\circ}(f, x)$.

(80) If F is commutative, then $F^{\circ}(f, g) = F^{\circ}(g, f)$.

(81) If F is idempotent, then $F^{\circ}(f, f) = f$.

(82) If F is idempotent, then $(F^{\circ}(f(y), f))(y) = f(y)$.

(83) If F is idempotent, then $(F^{\circ}(f, f(y)))(y) = f(y)$.

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