FORMALIZED MATHEMATICS
Number 1, January 1990
Université Catholique de Louvain

The Contraction Lemma

Grzegorz Bancerek[1]
Warsaw University
Białystok

Summary. The article includes the proof of the contraction lemma which claims that every class in which the axiom of extensionality is valid is isomorphic with a transitive class. In this article the isomorphism (wrt membership relation) of two sets is defined. It is based on [6].

The articles [7], [8], [4], [1], [5], [3], and [2] provide the terminology and notation for this paper. For simplicity we adopt the following convention: X, Y, Z denote objects of the type set; x, y denote objects of the type Any; E denotes an object of the type SET_DOMAIN; A, B, C denote objects of the type Ordinal; L denotes an object of the type Transfinite-Sequence; f denotes an object of the type Function; d, $d1$, d' denote objects of the type Element **of** E. Let us consider E, A. The functor

$$\mathrm{M}_{\mu}(E, A),$$

with values of the type set, is defined by

$$\mathbf{ex}\, L\ \mathbf{st\ it} = \{\, d : \mathbf{for}\, d1\ \mathbf{st}\ d1 \in d\ \mathbf{ex}\, B\ \mathbf{st}\ B \in \operatorname{dom} L\ \&\ d1 \in \bigcup\{L.B\}\,\}\ \&\ \operatorname{dom} L = A$$
$$\&\ \mathbf{for}\, B\ \mathbf{st}\ B \in A$$
$$\mathbf{holds}\ L.B = \{\, d1 : \mathbf{for}\, d\ \mathbf{st}\ d \in d1\ \mathbf{ex}\, C\ \mathbf{st}\ C \in \operatorname{dom}(L \mid B)\ \&\ d \in \bigcup\{L \mid B.C\}\,\}.$$

One can prove the following propositions:

(1) $$\mathrm{M}_{\mu}(E, A) = \{\, d : \mathbf{for}\, d1\ \mathbf{st}\ d1 \in d\ \mathbf{ex}\, B\ \mathbf{st}\ B \in A\ \&\ d1 \in \mathrm{M}_{\mu}(E, B)\,\},$$

(2) $$\mathbf{not}\,(\mathbf{ex}\, d1\ \mathbf{st}\ d1 \in d)\ \mathbf{iff}\ d \in \mathrm{M}_{\mu}(E, \mathbf{0}),$$

(3) $$d \cap E \subseteq \mathrm{M}_{\mu}(E, A)\ \mathbf{iff}\ d \in \mathrm{M}_{\mu}(E, \operatorname{succ} A),$$

(4) $$A \subseteq B\ \mathbf{implies}\ \mathrm{M}_{\mu}(E, A) \subseteq \mathrm{M}_{\mu}(E, B),$$

[1]Supported by RPBP.III-24.C1.

© 1990 Fondation Philippe le Hodey
ISSN 0777-4028

(5) $$\textbf{ex}\, A \;\textbf{st}\; d \in \mathrm{M}_\mu\,(E, A),$$

(6) $$d' \in d \;\&\; d \in \mathrm{M}_\mu\,(E, A)$$
$$\textbf{implies}\; d' \in \mathrm{M}_\mu\,(E, A) \;\&\; \textbf{ex}\, B \;\textbf{st}\; B \in A \;\&\; d' \in \mathrm{M}_\mu\,(E, B),$$

(7) $$\mathrm{M}_\mu\,(E, A) \subseteq E,$$

(8) $$\textbf{ex}\, A \;\textbf{st}\; E = \mathrm{M}_\mu\,(E, A),$$

(9) $$\textbf{ex}\, f \;\textbf{st}\; \mathrm{dom}\, f = E \;\&\; \textbf{for}\, d \;\textbf{holds}\; f\,.d = f^{\circ}\, d.$$

Let us consider f, X, Y. The predicate

$$f \text{ is\_}{\in}\text{-isomorphism\_of } X, Y$$

is defined by

$$\mathrm{dom}\, f = X \;\&\; \mathrm{rng}\, f = Y \;\&\; f \text{ is\_one-to-one} \;\&\; \textbf{for}\, x{,}y$$
$$\textbf{st}\; x \in X \;\&\; y \in X \;\textbf{holds}\; (\textbf{ex}\, Z \;\textbf{st}\; Z = y \;\&\; x \in Z) \;\textbf{iff}\; \textbf{ex}\, Z \;\textbf{st}\; f\,.y = Z \;\&\; f\,.x \in Z.$$

Next we state a proposition

(10) $$f \text{ is\_}{\in}\text{-isomorphism\_of } X, Y \;\textbf{iff}\; \mathrm{dom}\, f = X \;\&\; \mathrm{rng}\, f = Y \;\&\; f \text{ is\_one-to-one} \;\&$$
$$\textbf{for}\, x{,}y \;\textbf{st}\; x \in X \;\&\; y \in X$$
$$\textbf{holds}\; (\textbf{ex}\, Z \;\textbf{st}\; Z = y \;\&\; x \in Z) \;\textbf{iff}\; \textbf{ex}\, Z \;\textbf{st}\; f\,.y = Z \;\&\; f\,.x \in Z.$$

Let us consider X, Y. The predicate

X, Y are_$\in$-isomorphic   is defined by $\textbf{ex}\, f \;\textbf{st}\; f$ is_$\in$-isomorphism_of X, Y.

Next we state two propositions:

(11) $$X, Y \text{ are\_}{\in}\text{-isomorphic} \;\textbf{iff}\; \textbf{ex}\, f \;\textbf{st}\; f \text{ is\_}{\in}\text{-isomorphism\_of } X, Y,$$

(12) $$\mathrm{dom}\, f = E \;\&\; (\textbf{for}\, d \;\textbf{holds}\; f\,.d = f^{\circ}\, d) \;\textbf{implies}\; \mathrm{rng}\, f \text{ is\_}{\in}\text{-transitive}\,.$$

In the sequel u, v, w will denote objects of the type Element **of** E. Next we state two propositions:

(13) $$E \models \text{the\_axiom\_of\_extensionality}$$
$$\textbf{implies for}\; u{,}v \;\textbf{st for}\; w \;\textbf{holds}\; w \in u \;\textbf{iff}\; w \in v \;\textbf{holds}\; u = v,$$

(14) $$E \models \text{the\_axiom\_of\_extensionality}$$
$$\textbf{implies ex}\; X \;\textbf{st}\; X \text{ is\_}{\in}\text{-transitive} \;\&\; E, X \text{ are\_}{\in}\text{-isomorphic}\,.$$

References

[1] Grzegorz Bancerek. A model of ZF set theory language. *Formalized Mathematics*, 1, 1990.

[2] Grzegorz Bancerek. Models and satisfiability. *Formalized Mathematics*, 1, 1990.

[3] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1, 1990.

[4] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1, 1990.

[5] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1, 1990.

[6] Andrzej Mostowski. *Constructible Sets*. North Holland, 1970.

[7] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1, 1990.

[8] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1, 1990.

Received April 14, 1989