FORMALIZED MATHEMATICS
Number 1, January 1990
Université Catholique de Louvain

Subsets of Topological Spaces

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Summary. The article contains some theorems about open and closed sets. The following topological operations on sets are defined: closure, interior and frontier. The following notions are introduced: dense set, boundary set, nowheredense set and set being domain (closed domain and open domain), and some basic facts concerning them are proved.

The papers [4], [5], [3], [1], and [2] provide the notation and terminology for this paper. For simplicity we adopt the following convention: TS denotes an object of the type TopSpace; x denotes an object of the type Any; P, Q, G denote objects of the type Subset **of** TS; p denotes an object of the type Point **of** TS. One can prove the following propositions:

(1) $$x \in P \textbf{ implies } x \textbf{ is } \text{Point } \textbf{of } TS,$$

(2) $$P \cup \Omega\, TS = \Omega\, TS \;\&\; \Omega\, TS \cup P = \Omega\, TS,$$

(3) $$P \cap \Omega\, TS = P \;\&\; \Omega\, TS \cap P = P,$$

(4) $$P \cap \emptyset TS = \emptyset TS \;\&\; \emptyset TS \cap P = \emptyset TS,$$

(5) $$P^{\,\mathrm{c}} = \Omega\, TS \setminus P,$$

(6) $$P^{\,\mathrm{c}} = (P \textbf{ qua } \text{Subset } \textbf{of the } \text{carrier } \textbf{of } TS)^{\,\mathrm{c}},$$

(7) $$p \in P^{\,\mathrm{c}} \textbf{ iff not } p \in P,$$

(8) $$(\Omega\, TS)^{\,\mathrm{c}} = \emptyset TS,$$

(9) $$\Omega\, TS = (\emptyset TS)^{\,\mathrm{c}},$$

(10) $$(P^{\,\mathrm{c}})^{\,\mathrm{c}} = P,$$

[1]Supported by RPBP.III-24.C1.

ISSN 0777-4028

$$P \cup P^{\,\mathrm{c}} = \Omega TS \;\&\; P^{\,\mathrm{c}} \cup P = \Omega TS, \tag{11}$$

$$P \cap P^{\,\mathrm{c}} = \emptyset TS \;\&\; P^{\,\mathrm{c}} \cap P = \emptyset TS, \tag{12}$$

$$(P \cup Q)^{\,\mathrm{c}} = (P^{\,\mathrm{c}}) \cap (Q^{\,\mathrm{c}}), \tag{13}$$

$$(P \cap Q)^{\,\mathrm{c}} = (P^{\,\mathrm{c}}) \cup (Q^{\,\mathrm{c}}), \tag{14}$$

$$P \subseteq Q \textbf{ iff } Q^{\,\mathrm{c}} \subseteq P^{\,\mathrm{c}}, \tag{15}$$

$$P \setminus Q = P \cap Q^{\,\mathrm{c}}, \tag{16}$$

$$(P \setminus Q)^{\,\mathrm{c}} = P^{\,\mathrm{c}} \cup Q, \tag{17}$$

$$P \subseteq Q^{\,\mathrm{c}} \textbf{ implies } Q \subseteq P^{\,\mathrm{c}}, \tag{18}$$

$$P^{\,\mathrm{c}} \subseteq Q \textbf{ implies } Q^{\,\mathrm{c}} \subseteq P, \tag{19}$$

$$P \subseteq Q \textbf{ iff } P \cap Q^{\,\mathrm{c}} = \emptyset, \tag{20}$$

$$P^{\,\mathrm{c}} = Q^{\,\mathrm{c}} \textbf{ implies } P = Q, \tag{21}$$

$$\emptyset TS \,\mathrm{is\_closed}, \tag{22}$$

$$\mathrm{Cl}\,(\emptyset TS) = \emptyset TS, \tag{23}$$

$$P \subseteq \mathrm{Cl}\,P, \tag{24}$$

$$P \subseteq Q \textbf{ implies } \mathrm{Cl}\,P \subseteq \mathrm{Cl}\,Q, \tag{25}$$

$$\mathrm{Cl}\,(\mathrm{Cl}\,P) = \mathrm{Cl}\,P, \tag{26}$$

$$\mathrm{Cl}\,(\Omega TS) = \Omega TS, \tag{27}$$

$$\Omega TS \,\mathrm{is\_closed}, \tag{28}$$

$$P \,\mathrm{is\_closed} \textbf{ iff } P^{\,\mathrm{c}} \,\mathrm{is\_open}, \tag{29}$$

$$P \,\mathrm{is\_open} \textbf{ iff } P^{\,\mathrm{c}} \,\mathrm{is\_closed}, \tag{30}$$

$$Q \,\mathrm{is\_closed} \;\&\; P \subseteq Q \textbf{ implies } \mathrm{Cl}\,P \subseteq Q, \tag{31}$$

$$\mathrm{Cl}\,P \setminus \mathrm{Cl}\,Q \subseteq \mathrm{Cl}\,(P \setminus Q), \tag{32}$$

$$\mathrm{Cl}\,(P \cap Q) \subseteq \mathrm{Cl}\,P \cap \mathrm{Cl}\,Q, \tag{33}$$

$$P \,\mathrm{is\_closed} \;\&\; Q \,\mathrm{is\_closed} \textbf{ implies } \mathrm{Cl}\,(P \cap Q) = \mathrm{Cl}\,P \cap \mathrm{Cl}\,Q, \tag{34}$$

$$P \,\mathrm{is\_closed} \;\&\; Q \,\mathrm{is\_closed} \textbf{ implies } P \cap Q \,\mathrm{is\_closed}, \tag{35}$$

(36) $$P \text{ is\_closed } \& \ Q \text{ is\_closed } \textbf{implies } P \cup Q \text{ is\_closed},$$

(37) $$P \text{ is\_open } \& \ Q \text{ is\_open } \textbf{implies } P \cup Q \text{ is\_open},$$

(38) $$P \text{ is\_open } \& \ Q \text{ is\_open } \textbf{implies } P \cap Q \text{ is\_open},$$

(39) $$p \in \mathrm{Cl}\, P \textbf{ iff for } G \textbf{ st } G \text{ is\_open } \textbf{holds } p \in G \textbf{ implies } P \cap G \neq \emptyset,$$

(40) $$Q \text{ is\_open } \textbf{implies } Q \cap \mathrm{Cl}\, P \subseteq \mathrm{Cl}\,(Q \cap P),$$

(41) $$Q \text{ is\_open } \textbf{implies } \mathrm{Cl}\,(Q \cap \mathrm{Cl}\, P) = \mathrm{Cl}\,(Q \cap P).$$

Let us consider TS, P. The functor

$$\mathrm{Int}\, P,$$

yields the type Subset **of** TS and is defined by

$$\textbf{it} = (\mathrm{Cl}\,(P^{\,\mathrm{c}}))^{\,\mathrm{c}}.$$

One can prove the following propositions:

(42) $$\mathrm{Int}\, P = (\mathrm{Cl}\, P^{\,\mathrm{c}})^{\,\mathrm{c}},$$

(43) $$\mathrm{Int}\,(\Omega\, TS) = \Omega\, TS,$$

(44) $$\mathrm{Int}\, P \subseteq P,$$

(45) $$\mathrm{Int}\,(\mathrm{Int}\, P) = \mathrm{Int}\, P,$$

(46) $$\mathrm{Int}\, P \cap \mathrm{Int}\, Q = \mathrm{Int}\,(P \cap Q),$$

(47) $$\mathrm{Int}\,(\emptyset TS) = \emptyset TS,$$

(48) $$P \subseteq Q \textbf{ implies } \mathrm{Int}\, P \subseteq \mathrm{Int}\, Q,$$

(49) $$\mathrm{Int}\, P \cup \mathrm{Int}\, Q \subseteq \mathrm{Int}\,(P \cup Q),$$

(50) $$\mathrm{Int}\,(P \setminus Q) \subseteq \mathrm{Int}\, P \setminus \mathrm{Int}\, Q,$$

(51) $$\mathrm{Int}\, P \text{ is\_open},$$

(52) $$\emptyset TS \text{ is\_open},$$

(53) $$\Omega\, TS \text{ is\_open},$$

(54) $$x \in \mathrm{Int}\, P \textbf{ iff ex } Q \textbf{ st } Q \text{ is\_open } \& \ Q \subseteq P \ \& \ x \in Q,$$

(55) $$P \text{ is\_open } \textbf{iff } \mathrm{Int}\, P = P,$$

(56) $$Q \text{ is\_open } \& \ Q \subseteq P \textbf{ implies } Q \subseteq \mathrm{Int}\, P,$$

(57) $$P \text{ is\_open } \textbf{iff for } x \textbf{ holds } x \in P \textbf{ iff ex } Q \textbf{ st } Q \text{ is\_open } \& \ Q \subseteq P \ \& \ x \in Q,$$

(58) $$\mathrm{Cl}\,(\mathrm{Int}\,P) = \mathrm{Cl}\,(\mathrm{Int}\,(\mathrm{Cl}\,(\mathrm{Int}\,P))),$$

(59) $$P \text{ is\_open } \textbf{implies } \mathrm{Cl}\,(\mathrm{Int}\,(\mathrm{Cl}\,P)) = \mathrm{Cl}\,P.$$

Let us consider TS, P. The functor

$$\mathrm{Fr}\,P,$$

yields the type Subset **of** TS and is defined by

$$\textbf{it} = \mathrm{Cl}\,P \cap \mathrm{Cl}\,(P^{\mathrm{c}}).$$

We now state a number of propositions:

(60) $$\mathrm{Fr}\,P = \mathrm{Cl}\,P \cap \mathrm{Cl}\,(P^{\mathrm{c}}),$$

(61) $$p \in \mathrm{Fr}\,P \textbf{ iff for } Q \textbf{ st } Q \text{ is\_open } \& \ p \in Q \textbf{ holds } P \cap Q \neq \emptyset \ \& \ P^{\mathrm{c}} \cap Q \neq \emptyset,$$

(62) $$\mathrm{Fr}\,P = \mathrm{Fr}\,(P^{\mathrm{c}}),$$

(63) $$\mathrm{Fr}\,P \subseteq \mathrm{Cl}\,P,$$

(64) $$\mathrm{Fr}\,P = \mathrm{Cl}\,(P^{\mathrm{c}}) \cap P \cup (\mathrm{Cl}\,P \setminus P),$$

(65) $$\mathrm{Cl}\,P = P \cup \mathrm{Fr}\,P,$$

(66) $$\mathrm{Fr}\,(P \cap Q) \subseteq \mathrm{Fr}\,P \cup \mathrm{Fr}\,Q,$$

(67) $$\mathrm{Fr}\,(P \cup Q) \subseteq \mathrm{Fr}\,P \cup \mathrm{Fr}\,Q,$$

(68) $$\mathrm{Fr}\,(\mathrm{Fr}\,P) \subseteq \mathrm{Fr}\,P,$$

(69) $$P \text{ is\_closed } \textbf{implies } \mathrm{Fr}\,P \subseteq P,$$

(70) $$\mathrm{Fr}\,P \cup \mathrm{Fr}\,Q = \mathrm{Fr}\,(P \cup Q) \cup \mathrm{Fr}\,(P \cap Q) \cup (\mathrm{Fr}\,P \cap \mathrm{Fr}\,Q),$$

(71) $$\mathrm{Fr}\,(\mathrm{Int}\,P) \subseteq \mathrm{Fr}\,P,$$

(72) $$\mathrm{Fr}\,(\mathrm{Cl}\,P) \subseteq \mathrm{Fr}\,P,$$

(73) $$\mathrm{Int}\,P \cap \mathrm{Fr}\,P = \emptyset,$$

(74) $$\mathrm{Int}\,P = P \setminus \mathrm{Fr}\,P,$$

(75) $$\mathrm{Fr}\,(\mathrm{Fr}\,(\mathrm{Fr}\,P)) = \mathrm{Fr}\,(\mathrm{Fr}\,P),$$

(76) $$P \text{ is\_open } \textbf{iff } \mathrm{Fr}\,P = \mathrm{Cl}\,P \setminus P,$$

(77) $$P \text{ is\_closed } \textbf{iff } \mathrm{Fr}\,P = P \setminus \mathrm{Int}\,P.$$

Let us consider TS, P. The predicate

$$P\,\text{is\_dense} \qquad \text{is defined by} \qquad \mathrm{Cl}\,P = \Omega\,TS.$$

We now state several propositions:

(78) $$P\,\text{is\_dense}\ \textbf{iff}\ \mathrm{Cl}\,P = \Omega\,TS,$$

(79) $$P\,\text{is\_dense}\ \&\ P \subseteq Q\ \textbf{implies}\ Q\,\text{is\_dense}\,,$$

(80) $$P\,\text{is\_dense}\ \textbf{iff for}\ Q\ \textbf{st}\ Q \neq \emptyset\ \&\ Q\,\text{is\_open}\ \textbf{holds}\ P \cap Q \neq \emptyset,$$

(81) $$P\,\text{is\_dense}\ \textbf{implies for}\ Q\ \textbf{holds}\ Q\,\text{is\_open}\ \textbf{implies}\ \mathrm{Cl}\,Q = \mathrm{Cl}\,(Q \cap P),$$

(82) $$P\,\text{is\_dense}\ \&\ Q\,\text{is\_dense}\ \&\ Q\,\text{is\_open}\ \textbf{implies}\ P \cap Q\,\text{is\_dense}\,.$$

Let us consider TS, P. The predicate

$$P\,\text{is\_boundary} \qquad \text{is defined by} \qquad P^{\,\mathrm{c}}\,\text{is\_dense}\,.$$

Next we state several propositions:

(83) $$P\,\text{is\_boundary}\ \textbf{iff}\ P^{\,\mathrm{c}}\,\text{is\_dense}\,,$$

(84) $$P\,\text{is\_boundary}\ \textbf{iff}\ \mathrm{Int}\,P = \emptyset,$$

(85) $$P\,\text{is\_boundary}\ \&\ Q\,\text{is\_boundary}\ \&\ Q\,\text{is\_closed}\ \textbf{implies}\ P \cup Q\,\text{is\_boundary}\,,$$

(86) $$P\,\text{is\_boundary}\ \textbf{iff for}\ Q\ \textbf{st}\ Q \subseteq P\ \&\ Q\,\text{is\_open}\ \textbf{holds}\ Q = \emptyset,$$

(87) $$P\,\text{is\_closed}\ \textbf{implies}\ (P\,\text{is\_boundary}\ \textbf{iff for}\ Q$$
$$\textbf{st}\ Q \neq \emptyset\ \&\ Q\,\text{is\_open}\ \textbf{ex}\ G\ \textbf{st}\ G \subseteq Q\ \&\ G \neq \emptyset\ \&\ G\,\text{is\_open}\ \&\ P \cap G = \emptyset),$$

(88) $$P\,\text{is\_boundary}\ \textbf{iff}\ P \subseteq \mathrm{Fr}\,P.$$

Let us consider TS, P. The predicate

$$P\,\text{is\_nowheredense} \qquad \text{is defined by} \qquad \mathrm{Cl}\,P\,\text{is\_boundary}\,.$$

One can prove the following propositions:

(89) $$P\,\text{is\_nowheredense}\ \textbf{iff}\ \mathrm{Cl}\,P\,\text{is\_boundary}\,,$$

(90) $$P\,\text{is\_nowheredense}\ \&\ Q\,\text{is\_nowheredense}\ \textbf{implies}\ P \cup Q\,\text{is\_nowheredense}\,,$$

(91) $$P\,\text{is\_nowheredense}\ \textbf{implies}\ P^{\,\mathrm{c}}\,\text{is\_dense}\,,$$

(92) $$P\,\text{is\_nowheredense}\ \textbf{implies}\ P\,\text{is\_boundary}\,,$$

(93) $$Q\,\text{is\_boundary}\ \&\ Q\,\text{is\_closed}\ \textbf{implies}\ Q\,\text{is\_nowheredense}\,,$$

(94) P is_closed **implies** (P is_nowheredense **iff** $P = \mathrm{Fr}\, P$),

(95) P is_open **implies** $\mathrm{Fr}\, P$ is_nowheredense ,

(96) P is_closed **implies** $\mathrm{Fr}\, P$ is_nowheredense ,

(97) P is_open & P is_nowheredense **implies** $P = \emptyset$.

We now define three new predicates. Let us consider TS, P. The predicate

P is_domain is defined by $\mathrm{Int}\,(\mathrm{Cl}\, P) \subseteq P$ & $P \subseteq \mathrm{Cl}\,(\mathrm{Int}\, P)$.

The predicate

P is_closed_domain is defined by $P = \mathrm{Cl}\,(\mathrm{Int}\, P)$.

The predicate

P is_open_domain is defined by $P = \mathrm{Int}\,(\mathrm{Cl}\, P)$.

The following propositions are true:

(98) P is_domain **iff** $\mathrm{Int}\,(\mathrm{Cl}\, P) \subseteq P$ & $P \subseteq \mathrm{Cl}\,(\mathrm{Int}\, P)$,

(99) P is_closed_domain **iff** $P = \mathrm{Cl}\,(\mathrm{Int}\, P)$,

(100) P is_open_domain **iff** $P = \mathrm{Int}\,(\mathrm{Cl}\, P)$,

(101) P is_open_domain **iff** P^{c} is_closed_domain ,

(102) P is_closed_domain **implies** $\mathrm{Fr}\,(\mathrm{Int}\, P) = \mathrm{Fr}\, P$,

(103) P is_closed_domain **implies** $\mathrm{Fr}\, P \subseteq \mathrm{Cl}\,(\mathrm{Int}\, P)$,

(104) P is_open_domain **implies** $\mathrm{Fr}\, P = \mathrm{Fr}\,(\mathrm{Cl}\, P)$ & $\mathrm{Fr}\,(\mathrm{Cl}\, P) = \mathrm{Cl}\, P \setminus P$,

(105) P is_open & P is_closed **implies** (P is_closed_domain **iff** P is_open_domain),

(106) P is_closed & P is_domain **iff** P is_closed_domain ,

(107) P is_open & P is_domain **iff** P is_open_domain ,

(108) P is_closed_domain & Q is_closed_domain **implies** $P \cup Q$ is_closed_domain ,

(109) P is_open_domain & Q is_open_domain **implies** $P \cap Q$ is_open_domain ,

(110) P is_domain **implies** $\mathrm{Int}\,(\mathrm{Fr}\, P) = \emptyset$,

(111) P is_domain **implies** $\mathrm{Int}\, P$ is_domain & $\mathrm{Cl}\, P$ is_domain .

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Received April 28, 1989