FORMALIZED MATHEMATICS
Number 1, January 1990
Université Catholique de Louvain

Properties of Binary Relations

Edmund Woronowicz[1]
Warsaw University
Białystok

Anna Zalewska[2]
Warsaw University
Białystok

Summary. The paper contains definitions of some properties of binary relations: reflexivity, irreflexivity, symmetry, asymmetry, antisymmetry, connectedness, strong connectedness, and transitivity. Basic theorems relating the above mentioned notions are given.

The terminology and notation used here have been introduced in the following articles: [1], [2], and [3]. For simplicity we adopt the following convention: X will have the type set; x, y, z will have the type Any; P, R will have the type Relation. We now define several new predicates. Let us consider R, X. The predicate

R is_reflexive_in X is defined by $x \in X$ **implies** $\langle x, x\rangle \in R$.

The predicate

R is_irreflexive_in X is defined by $x \in X$ **implies not** $\langle x, x\rangle \in R$.

The predicate

R is_symmetric_in X

is defined by

$x \in X$ & $y \in X$ & $\langle x, y\rangle \in R$ **implies** $\langle y, x\rangle \in R$.

The predicate

R is_antisymmetric_in X

is defined by

$x \in X$ & $y \in X$ & $\langle x, y\rangle \in R$ & $\langle y, x\rangle \in R$ **implies** $x = y$.

[1]Supported by RPBP III.24 C1.
[2]Supported by RPBP III.24 C1.

ISSN 0777-4028

The predicate

$$R \text{ is\_asymmetric\_in } X$$

is defined by

$$x \in X \;\&\; y \in X \;\&\; \langle x, y\rangle \in R \textbf{ implies not } \langle y, x\rangle \in R.$$

The predicate

$$R \text{ is\_connected\_in } X$$

is defined by

$$x \in X \;\&\; y \in X \;\&\; x \neq y \textbf{ implies } \langle x, y\rangle \in R \textbf{ or } \langle y, x\rangle \in R.$$

The predicate

$$R \text{ is\_strongly\_connected\_in } X$$

is defined by

$$x \in X \;\&\; y \in X \textbf{ implies } \langle x, y\rangle \in R \textbf{ or } \langle y, x\rangle \in R.$$

The predicate

$$R \text{ is\_transitive\_in } X$$

is defined by

$$x \in X \;\&\; y \in X \;\&\; z \in X \;\&\; \langle x, y\rangle \in R \;\&\; \langle y, z\rangle \in R \textbf{ implies } \langle x, z\rangle \in R.$$

We now state several propositions:

(1) $$R \text{ is\_reflexive\_in } X \textbf{ iff for } x \textbf{ st } x \in X \textbf{ holds } \langle x, x\rangle \in R,$$

(2) $$R \text{ is\_irreflexive\_in } X \textbf{ iff for } x \textbf{ st } x \in X \textbf{ holds not } \langle x, x\rangle \in R,$$

(3) $$R \text{ is\_symmetric\_in } X$$
$$\textbf{iff for } x,y \textbf{ st } x \in X \;\&\; y \in X \;\&\; \langle x, y\rangle \in R \textbf{ holds } \langle y, x\rangle \in R,$$

(4) $$R \text{ is\_antisymmetric\_in } X$$
$$\textbf{iff for } x,y \textbf{ st } x \in X \;\&\; y \in X \;\&\; \langle x, y\rangle \in R \;\&\; \langle y, x\rangle \in R \textbf{ holds } x = y,$$

(5) $$R \text{ is\_asymmetric\_in } X$$
$$\textbf{iff for } x,y \textbf{ st } x \in X \;\&\; y \in X \;\&\; \langle x, y\rangle \in R \textbf{ holds not } \langle y, x\rangle \in R,$$

(6) $$R \text{ is\_connected\_in } X$$
$$\textbf{iff for } x,y \textbf{ st } x \in X \;\&\; y \in X \;\&\; x \neq y \textbf{ holds } \langle x, y\rangle \in R \textbf{ or } \langle y, x\rangle \in R,$$

(7) $$R \text{ is\_strongly\_connected\_in } X$$
$$\textbf{iff for } x,y \textbf{ st } x \in X \;\&\; y \in X \textbf{ holds } \langle x, y\rangle \in R \textbf{ or } \langle y, x\rangle \in R,$$

(8) R is_transitive_in X **iff** **for** x,y,z
st $x \in X$ & $y \in X$ & $z \in X$ & $\langle x, y\rangle \in R$ & $\langle y, z\rangle \in R$ **holds** $\langle x, z\rangle \in R$.

We now define several new predicates. Let us consider R. The predicate

R is_reflexive is defined by R is_reflexive_in field R.

The predicate

R is_irreflexive is defined by R is_irreflexive_in field R.

The predicate

R is_symmetric is defined by R is_symmetric_in field R.

The predicate

R is_antisymmetric is defined by R is_antisymmetric_in field R.

The predicate

R is_asymmetric is defined by R is_asymmetric_in field R.

The predicate

R is_connected is defined by R is_connected_in field R.

The predicate

R is_strongly_connected is defined by R is_strongly_connected_in field R.

The predicate

R is_transitive is defined by R is_transitive_in field R.

We now state a number of propositions:

(9) R is_reflexive **iff** R is_reflexive_in field R,

(10) R is_irreflexive **iff** R is_irreflexive_in field R,

(11) R is_symmetric **iff** R is_symmetric_in field R,

(12) R is_antisymmetric **iff** R is_antisymmetric_in field R,

(13) R is_asymmetric **iff** R is_asymmetric_in field R,

(14) R is_connected **iff** R is_connected_in field R,

(15) R is_strongly_connected **iff** R is_strongly_connected_in field R,

(16) R is_transitive **iff** R is_transitive_in field R,

$$R \text{ is\_reflexive } \textbf{iff } \triangle \operatorname{field} R \subseteq R, \tag{17}$$

$$R \text{ is\_irreflexive } \textbf{iff } \triangle (\operatorname{field} R) \cap R = \emptyset, \tag{18}$$

$$R \text{ is\_antisymmetric\_in } X \textbf{ iff } R \setminus \triangle X \text{ is\_asymmetric\_in } X, \tag{19}$$

$$R \text{ is\_asymmetric\_in } X \textbf{ implies } R \cup \triangle X \text{ is\_antisymmetric\_in } X, \tag{20}$$

$$R \text{ is\_antisymmetric\_in } X \textbf{ implies } R \setminus \triangle X \text{ is\_asymmetric\_in } X, \tag{21}$$

$$R \text{ is\_symmetric } \& \ R \text{ is\_transitive } \textbf{implies } R \text{ is\_reflexive}, \tag{22}$$

$$\triangle X \text{ is\_symmetric } \& \triangle X \text{ is\_transitive}, \tag{23}$$

$$\triangle X \text{ is\_antisymmetric } \& \triangle X \text{ is\_reflexive}, \tag{24}$$

$$R \text{ is\_irreflexive } \& \ R \text{ is\_transitive } \textbf{implies } R \text{ is\_asymmetric}, \tag{25}$$

$$R \text{ is\_asymmetric } \textbf{implies } R \text{ is\_irreflexive } \& \ R \text{ is\_antisymmetric}, \tag{26}$$

$$R \text{ is\_reflexive } \textbf{implies } R\tilde{} \text{ is\_reflexive}, \tag{27}$$

$$R \text{ is\_irreflexive } \textbf{implies } R\tilde{} \text{ is\_irreflexive}, \tag{28}$$

$$R \text{ is\_reflexive } \textbf{implies } \operatorname{dom} R = \operatorname{dom}(R\tilde{}) \ \& \ \operatorname{rng} R = \operatorname{rng}(R\tilde{}), \tag{29}$$

$$R \text{ is\_symmetric } \textbf{iff } R = R\tilde{}, \tag{30}$$

$$P \text{ is\_reflexive } \& \ R \text{ is\_reflexive } \textbf{implies } P \cup R \text{ is\_reflexive } \& \ P \cap R \text{ is\_reflexive}, \tag{31}$$

$$\begin{gathered} P \text{ is\_irreflexive } \& \ R \text{ is\_irreflexive} \\ \textbf{implies } P \cup R \text{ is\_irreflexive } \& \ P \cap R \text{ is\_irreflexive}, \end{gathered} \tag{32}$$

$$P \text{ is\_irreflexive } \textbf{implies } P \setminus R \text{ is\_irreflexive}, \tag{33}$$

$$R \text{ is\_symmetric } \textbf{implies } R\tilde{} \text{ is\_symmetric}, \tag{34}$$

$$\begin{gathered} P \text{ is\_symmetric } \& \ R \text{ is\_symmetric} \\ \textbf{implies } P \cup R \text{ is\_symmetric } \& \ P \cap R \text{ is\_symmetric } \& \ P \setminus R \text{ is\_symmetric}, \end{gathered} \tag{35}$$

$$R \text{ is\_asymmetric } \textbf{implies } R\tilde{} \text{ is\_asymmetric}, \tag{36}$$

$$P \text{ is\_asymmetric } \& \ R \text{ is\_asymmetric } \textbf{implies } P \cap R \text{ is\_asymmetric}, \tag{37}$$

$$P \text{ is\_asymmetric } \textbf{implies } P \setminus R \text{ is\_asymmetric}, \tag{38}$$

$$R \text{ is\_antisymmetric } \textbf{iff } R \cap (R\tilde{}) \subseteq \triangle (\operatorname{dom} R), \tag{39}$$

$$R \text{ is\_antisymmetric } \textbf{implies } R\tilde{} \text{ is\_antisymmetric}, \tag{40}$$

(41) $$P \text{ is\_antisymmetric}$$
$$\textbf{implies } P \cap R \text{ is\_antisymmetric } \& \ P \setminus R \text{ is\_antisymmetric},$$

(42) $$R \text{ is\_transitive } \textbf{implies } R\tilde{}\ \text{is\_transitive},$$

(43) $$P \text{ is\_transitive } \& \ R \text{ is\_transitive } \textbf{implies } P \cap R \text{ is\_transitive},$$

(44) $$R \text{ is\_transitive } \textbf{iff } R \cdot R \subseteq R,$$

(45) $$R \text{ is\_connected } \textbf{iff } [:\text{field } R, \text{field } R:] \setminus \triangle(\text{field } R) \subseteq R \cup R\tilde{},$$

(46) $$R \text{ is\_strongly\_connected } \textbf{implies } R \text{ is\_connected } \& \ R \text{ is\_reflexive},$$

(47) $$R \text{ is\_strongly\_connected } \textbf{iff } [:\text{field } R, \text{field } R:] = R \cup R\tilde{}.$$

References

[1] Andrzej Trybulec. Tarski Grothendieck set theory. *Formalized Mathematics*, 1, 1990.

[2] Zinaida Trybulec and Halina Święczkowska. Boolean properties of sets. *Formalized Mathematics*, 1, 1990.

[3] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1, 1990.

Received March 15, 1989