JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
Inst. of Computer Science, Univ. of Białystok

Replacing of Variables in Formulas of ZF Theory

Grzegorz Bancerek
Warsaw University
Białystok

Summary. Part one is a supplement to papers [1], [2], and [3]. It deals with concepts of selector functions, atomic, negative, conjunctive formulas and etc., subformulas, free variables, satisfiability and models (it is shown that axioms of the predicate and the quantifier calculus are satisfied in an arbitrary set). In part two there are introduced notions of variables occurring in a formula and replacing of variables in a formula.

MML Identifier: ZF_LANG1.

WWW: http://mizar.org/JFM/Vol2/zf_lang1.html

The articles [10], [9], [7], [12], [11], [13], [5], [6], [4], [8], [1], and [2] provide the notation and terminology for this paper.

For simplicity, we adopt the following convention: p, p_1, p_2, q, r, F, G, G_1, G_2, H, H_1, H_2 are ZF-formulae, x, x_1, x_2, y, y_1, y_2, z, z_1, z_2, s, t are variables, a is a set, and X is a set.

The following propositions are true:

(1) $\mathrm{Var}_1(x{=}y) = x$ and $\mathrm{Var}_2(x{=}y) = y$.

(2) $\mathrm{Var}_1(x \varepsilon y) = x$ and $\mathrm{Var}_2(x \varepsilon y) = y$.

(3) $\mathrm{Arg}(\neg p) = p$.

(4) $\mathrm{LeftArg}(p \wedge q) = p$ and $\mathrm{RightArg}(p \wedge q) = q$.

(5) $\mathrm{LeftArg}(p \vee q) = p$ and $\mathrm{RightArg}(p \vee q) = q$.

(6) $\mathrm{Antecedent}(p \Rightarrow q) = p$ and $\mathrm{Consequent}(p \Rightarrow q) = q$.

(7) $\mathrm{LeftSide}(p \Leftrightarrow q) = p$ and $\mathrm{RightSide}(p \Leftrightarrow q) = q$.

(8) $\mathrm{Bound}(\forall_x p) = x$ and $\mathrm{Scope}(\forall_x p) = p$.

(9) $\mathrm{Bound}(\exists_x p) = x$ and $\mathrm{Scope}(\exists_x p) = p$.

(10) $p \vee q = \neg p \Rightarrow q$.

(11) If $\forall_{x,y} p = \forall_z q$, then $x = z$ and $\forall_y p = q$.

(12) If $\exists_{x,y} p = \exists_z q$, then $x = z$ and $\exists_y p = q$.

(13) $\forall_{x,y} p$ is universal and $\mathrm{Bound}(\forall_{x,y} p) = x$ and $\mathrm{Scope}(\forall_{x,y} p) = \forall_y p$.

(14) $\exists_{x,y} p$ is existential and $\mathrm{Bound}(\exists_{x,y} p) = x$ and $\mathrm{Scope}(\exists_{x,y} p) = \exists_y p$.

(15) $\forall_{x,y,z} p = \forall_x \forall_y \forall_z p$ and $\forall_{x,y,z} p = \forall_{x,y} \forall_z p$.

(16) If $\forall_{x_1,y_1} p_1 = \forall_{x_2,y_2} p_2$, then $x_1 = x_2$ and $y_1 = y_2$ and $p_1 = p_2$.

(17) If $\forall_{x_1,y_1,z_1} p_1 = \forall_{x_2,y_2,z_2} p_2$, then $x_1 = x_2$ and $y_1 = y_2$ and $z_1 = z_2$ and $p_1 = p_2$.

(18) If $\forall_{x,y,z} p = \forall_t q$, then $x = t$ and $\forall_{y,z} p = q$.

(19) If $\forall_{x,y,z} p = \forall_{t,s} q$, then $x = t$ and $y = s$ and $\forall_z p = q$.

(20) If $\exists_{x_1,y_1} p_1 = \exists_{x_2,y_2} p_2$, then $x_1 = x_2$ and $y_1 = y_2$ and $p_1 = p_2$.

(21) $\exists_{x,y,z} p = \exists_x \exists_y \exists_z p$ and $\exists_{x,y,z} p = \exists_{x,y} \exists_z p$.

(22) If $\exists_{x_1,y_1,z_1} p_1 = \exists_{x_2,y_2,z_2} p_2$, then $x_1 = x_2$ and $y_1 = y_2$ and $z_1 = z_2$ and $p_1 = p_2$.

(23) If $\exists_{x,y,z} p = \exists_t q$, then $x = t$ and $\exists_{y,z} p = q$.

(24) If $\exists_{x,y,z} p = \exists_{t,s} q$, then $x = t$ and $y = s$ and $\exists_z p = q$.

(25) $\forall_{x,y,z} p$ is universal and $\mathrm{Bound}(\forall_{x,y,z} p) = x$ and $\mathrm{Scope}(\forall_{x,y,z} p) = \forall_{y,z} p$.

(26) $\exists_{x,y,z} p$ is existential and $\mathrm{Bound}(\exists_{x,y,z} p) = x$ and $\mathrm{Scope}(\exists_{x,y,z} p) = \exists_{y,z} p$.

(27) If H is disjunctive, then $\mathrm{LeftArg}(H) = \mathrm{Arg}(\mathrm{LeftArg}(\mathrm{Arg}(H)))$.

(28) If H is disjunctive, then $\mathrm{RightArg}(H) = \mathrm{Arg}(\mathrm{RightArg}(\mathrm{Arg}(H)))$.

(29) If H is conditional, then $\mathrm{Antecedent}(H) = \mathrm{LeftArg}(\mathrm{Arg}(H))$.

(30) If H is conditional, then $\mathrm{Consequent}(H) = \mathrm{Arg}(\mathrm{RightArg}(\mathrm{Arg}(H)))$.

(31) If H is biconditional, then $\mathrm{LeftSide}(H) = \mathrm{Antecedent}(\mathrm{LeftArg}(H))$ and $\mathrm{LeftSide}(H) = \mathrm{Consequent}(\mathrm{RightArg}(H))$.

(32) If H is biconditional, then $\mathrm{RightSide}(H) = \mathrm{Consequent}(\mathrm{LeftArg}(H))$ and $\mathrm{RightSide}(H) = \mathrm{Antecedent}(\mathrm{RightArg}(H))$.

(33) If H is existential, then $\mathrm{Bound}(H) = \mathrm{Bound}(\mathrm{Arg}(H))$ and $\mathrm{Scope}(H) = \mathrm{Arg}(\mathrm{Scope}(\mathrm{Arg}(H)))$.

(34) $\mathrm{Arg}(F \vee G) = \neg F \wedge \neg G$ and $\mathrm{Antecedent}(F \vee G) = \neg F$ and $\mathrm{Consequent}(F \vee G) = G$.

(35) $\mathrm{Arg}(F \Rightarrow G) = F \wedge \neg G$.

(36) $\mathrm{LeftArg}(F \Leftrightarrow G) = F \Rightarrow G$ and $\mathrm{RightArg}(F \Leftrightarrow G) = G \Rightarrow F$.

(37) $\mathrm{Arg}(\exists_x H) = \forall_x \neg H$.

(38) Suppose H is disjunctive. Then H is conditional and negative and $\mathrm{Arg}(H)$ is conjunctive and $\mathrm{LeftArg}(\mathrm{Arg}(H))$ is negative and $\mathrm{RightArg}(\mathrm{Arg}(H))$ is negative.

(39) If H is conditional, then H is negative and $\mathrm{Arg}(H)$ is conjunctive and $\mathrm{RightArg}(\mathrm{Arg}(H))$ is negative.

(40) If H is biconditional, then H is conjunctive and $\mathrm{LeftArg}(H)$ is conditional and $\mathrm{RightArg}(H)$ is conditional.

(41) If H is existential, then H is negative and $\mathrm{Arg}(H)$ is universal and $\mathrm{Scope}(\mathrm{Arg}(H))$ is negative.

(42) H is an equality, a membership, negative, conjunctive, universal, a membership, negative, conjunctive, universal, negative, conjunctive, universal, conjunctive, and universal.

(43) If F is a subformula of G, then $\mathrm{len}\, F \leq \mathrm{len}\, G$.

(44) Suppose that

(i) F is a proper subformula of G and G is a subformula of H, or

(ii) F is a subformula of G and G is a proper subformula of H, or

(iii) F is a subformula of G and G is an immediate constituent of H, or

(iv) F is an immediate constituent of G and G is a subformula of H, or

(v) F is a proper subformula of G and G is an immediate constituent of H, or

(vi) F is an immediate constituent of G and G is a proper subformula of H.

Then F is a proper subformula of H.

(46)[1] H is not an immediate constituent of H.

(47) G is not a proper subformula of H or H is not a subformula of G.

(48) G is not a proper subformula of H or H is not a proper subformula of G.

(49) G is not a subformula of H or H is not an immediate constituent of G.

(50) G is not a proper subformula of H or H is not an immediate constituent of G.

(51) If $\neg F$ is a subformula of H, then F is a proper subformula of H.

(52) If $F \wedge G$ is a subformula of H, then F is a proper subformula of H and G is a proper subformula of H.

(53) If $\forall_x H$ is a subformula of F, then H is a proper subformula of F.

(54)(i) $F \wedge \neg G$ is a proper subformula of $F \Rightarrow G$,

(ii) F is a proper subformula of $F \Rightarrow G$,

(iii) $\neg G$ is a proper subformula of $F \Rightarrow G$, and

(iv) G is a proper subformula of $F \Rightarrow G$.

(55)(i) $\neg F \wedge \neg G$ is a proper subformula of $F \vee G$,

(ii) $\neg F$ is a proper subformula of $F \vee G$,

(iii) $\neg G$ is a proper subformula of $F \vee G$,

(iv) F is a proper subformula of $F \vee G$, and

(v) G is a proper subformula of $F \vee G$.

(56) $\forall_x \neg H$ is a proper subformula of $\exists_x H$ and $\neg H$ is a proper subformula of $\exists_x H$.

(57) G is a subformula of H iff $G \in \mathrm{Subformulae} H$.

(58) If $G \in \mathrm{Subformulae} H$, then $\mathrm{Subformulae} G \subseteq \mathrm{Subformulae} H$.

(59) $H \in \mathrm{Subformulae} H$.

(60) $\mathrm{Subformulae}(F \Rightarrow G) = \mathrm{Subformulae} F \cup \mathrm{Subformulae} G \cup \{\neg G, F \wedge \neg G, F \Rightarrow G\}$.

(61) $\mathrm{Subformulae}(F \vee G) = \mathrm{Subformulae} F \cup \mathrm{Subformulae} G \cup \{\neg G, \neg F, \neg F \wedge \neg G, F \vee G\}$.

(62) $\mathrm{Subformulae}(F \Leftrightarrow G) = \mathrm{Subformulae} F \cup \mathrm{Subformulae} G \cup \{\neg G, F \wedge \neg G, F \Rightarrow G, \neg F, G \wedge \neg F, G \Rightarrow F, F \Leftrightarrow G\}$.

(63) $\mathrm{Free}\, x{=}y = \{x, y\}$.

(64) $\mathrm{Free}\, x \varepsilon y = \{x, y\}$.

(65) $\mathrm{Free}\, \neg p = \mathrm{Free}\, p$.

[1] The proposition (45) has been removed.

(66) $\mathrm{Free}(p \wedge q) = \mathrm{Free}\, p \cup \mathrm{Free}\, q$.

(67) $\mathrm{Free}\, \forall_x p = \mathrm{Free}\, p \setminus \{x\}$.

(68) $\mathrm{Free}(p \vee q) = \mathrm{Free}\, p \cup \mathrm{Free}\, q$.

(69) $\mathrm{Free}(p \Rightarrow q) = \mathrm{Free}\, p \cup \mathrm{Free}\, q$.

(70) $\mathrm{Free}(p \Leftrightarrow q) = \mathrm{Free}\, p \cup \mathrm{Free}\, q$.

(71) $\mathrm{Free}\, \exists_x p = \mathrm{Free}\, p \setminus \{x\}$.

(72) $\mathrm{Free}\, \forall_{x,y} p = \mathrm{Free}\, p \setminus \{x, y\}$.

(73) $\mathrm{Free}\, \forall_{x,y,z} p = \mathrm{Free}\, p \setminus \{x, y, z\}$.

(74) $\mathrm{Free}\, \exists_{x,y} p = \mathrm{Free}\, p \setminus \{x, y\}$.

(75) $\mathrm{Free}\, \exists_{x,y,z} p = \mathrm{Free}\, p \setminus \{x, y, z\}$.

The scheme *ZF Induction* concerns a unary predicate $\mathcal{P}$, and states that:
For every H holds $\mathcal{P}[H]$
provided the parameters meet the following conditions:

- For all x_1, x_2 holds $\mathcal{P}[x_1 = x_2]$ and $\mathcal{P}[x_1 \varepsilon x_2]$,
- For every H such that $\mathcal{P}[H]$ holds $\mathcal{P}[\neg H]$,
- For all H_1, H_2 such that $\mathcal{P}[H_1]$ and $\mathcal{P}[H_2]$ holds $\mathcal{P}[H_1 \wedge H_2]$, and
- For all H, x such that $\mathcal{P}[H]$ holds $\mathcal{P}[\forall_x H]$.

For simplicity, we use the following convention: M, E are non empty sets, e is an element of E, m, m' are elements of M, f is a function from VAR into E, and v, v' are functions from VAR into M.

Let us consider E, f, x, e. The functor $f(\frac{x}{e})$ yields a function from VAR into E and is defined by:

(Def. 1) $f(\frac{x}{e})(x) = e$ and for every y such that $f(\frac{x}{e})(y) \neq f(y)$ holds $x = y$.

Let D, D_1, D_2 be non empty sets and let f be a function from D into D_1. Let us assume that $D_1 \subseteq D_2$. The functor $D_2[f]$ yields a function from D into D_2 and is defined as follows:

(Def. 2) $D_2[f] = f$.

Next we state several propositions:

(78)[2] $v(\frac{x}{m'})(\frac{x}{m}) = v(\frac{x}{m})$ and $v(\frac{x}{v(x)}) = v$.

(79) If $x \neq y$, then $v(\frac{x}{m})(\frac{y}{m'}) = v(\frac{y}{m'})(\frac{x}{m})$.

(80) $M, v \models \forall_x H$ iff for every m holds $M, v(\frac{x}{m}) \models H$.

(81) $M, v \models \forall_x H$ iff $M, v(\frac{x}{m}) \models \forall_x H$.

(82) $M, v \models \exists_x H$ iff there exists m such that $M, v(\frac{x}{m}) \models H$.

(83) $M, v \models \exists_x H$ iff $M, v(\frac{x}{m}) \models \exists_x H$.

(84) For all v, v' such that for every x such that $x \in \mathrm{Free}\, H$ holds $v'(x) = v(x)$ holds if $M, v \models H$, then $M, v' \models H$.

(85) $\mathrm{Free}\, H$ is finite.

Let us consider H. One can verify that $\mathrm{Free}\, H$ is finite.
In the sequel i, j denote natural numbers.
The following propositions are true:

[2] The propositions (76) and (77) have been removed.

(86) If $\mathrm{x}_i = \mathrm{x}_j$, then $i = j$.

(87) There exists i such that $x = \mathrm{x}_i$.

(89)[3] $M, v \models x{=}x$.

(90) $M \models x{=}x$.

(91) $M, v \not\models x\varepsilon x$.

(92) $M \not\models x\varepsilon x$ and $M \models \neg x\varepsilon x$.

(93) $M \models x{=}y$ iff $x = y$ or there exists a such that $\{a\} = M$.

(94) $M \models \neg x\varepsilon y$ iff $x = y$ or for every X such that $X \in M$ holds X misses M.

(95) If H is an equality, then $M, v \models H$ iff $v(\mathrm{Var}_1(H)) = v(\mathrm{Var}_2(H))$.

(96) If H is a membership, then $M, v \models H$ iff $v(\mathrm{Var}_1(H)) \in v(\mathrm{Var}_2(H))$.

(97) If H is negative, then $M, v \models H$ iff $M, v \not\models \mathrm{Arg}(H)$.

(98) If H is conjunctive, then $M, v \models H$ iff $M, v \models \mathrm{LeftArg}(H)$ and $M, v \models \mathrm{RightArg}(H)$.

(99) If H is universal, then $M, v \models H$ iff for every m holds $M, v(\frac{\mathrm{Bound}(H)}{m}) \models \mathrm{Scope}(H)$.

(100) If H is disjunctive, then $M, v \models H$ iff $M, v \models \mathrm{LeftArg}(H)$ or $M, v \models \mathrm{RightArg}(H)$.

(101) If H is conditional, then $M, v \models H$ iff if $M, v \models \mathrm{Antecedent}(H)$, then $M, v \models \mathrm{Consequent}(H)$.

(102) If H is biconditional, then $M, v \models H$ iff $M, v \models \mathrm{LeftSide}(H)$ iff $M, v \models \mathrm{RightSide}(H)$.

(103) If H is existential, then $M, v \models H$ iff there exists m such that $M, v(\frac{\mathrm{Bound}(H)}{m}) \models \mathrm{Scope}(H)$.

(104) $M \models \exists_x H$ iff for every v there exists m such that $M, v(\frac{x}{m}) \models H$.

(105) If $M \models H$, then $M \models \exists_x H$.

(106) $M \models H$ iff $M \models \forall_{x,y} H$.

(107) If $M \models H$, then $M \models \exists_{x,y} H$.

(108) $M \models H$ iff $M \models \forall_{x,y,z} H$.

(109) If $M \models H$, then $M \models \exists_{x,y,z} H$.

(110) $M, v \models (p \Leftrightarrow q) \Rightarrow (p \Rightarrow q)$ and $M \models (p \Leftrightarrow q) \Rightarrow (p \Rightarrow q)$.

(111) $M, v \models (p \Leftrightarrow q) \Rightarrow (q \Rightarrow p)$ and $M \models (p \Leftrightarrow q) \Rightarrow (q \Rightarrow p)$.

(112) $M \models (p \Rightarrow q) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \Rightarrow r))$.

(113) If $M, v \models p \Rightarrow q$ and $M, v \models q \Rightarrow r$, then $M, v \models p \Rightarrow r$.

(114) If $M \models p \Rightarrow q$ and $M \models q \Rightarrow r$, then $M \models p \Rightarrow r$.

(115) $M, v \models (p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$ and $M \models (p \Rightarrow q) \wedge (q \Rightarrow r) \Rightarrow (p \Rightarrow r)$.

(116) $M, v \models p \Rightarrow (q \Rightarrow p)$ and $M \models p \Rightarrow (q \Rightarrow p)$.

(117) $M, v \models (p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$ and $M \models (p \Rightarrow (q \Rightarrow r)) \Rightarrow ((p \Rightarrow q) \Rightarrow (p \Rightarrow r))$.

(118) $M, v \models p \wedge q \Rightarrow p$ and $M \models p \wedge q \Rightarrow p$.

[3] The proposition (88) has been removed.

(119) $M, v \models p \wedge q \Rightarrow q$ and $M \models p \wedge q \Rightarrow q$.

(120) $M, v \models p \wedge q \Rightarrow q \wedge p$ and $M \models p \wedge q \Rightarrow q \wedge p$.

(121) $M, v \models p \Rightarrow p \wedge p$ and $M \models p \Rightarrow p \wedge p$.

(122) $M, v \models (p \Rightarrow q) \Rightarrow ((p \Rightarrow r) \Rightarrow (p \Rightarrow q \wedge r))$ and $M \models (p \Rightarrow q) \Rightarrow ((p \Rightarrow r) \Rightarrow (p \Rightarrow q \wedge r))$.

(123) $M, v \models p \Rightarrow p \vee q$ and $M \models p \Rightarrow p \vee q$.

(124) $M, v \models q \Rightarrow p \vee q$ and $M \models q \Rightarrow p \vee q$.

(125) $M, v \models p \vee q \Rightarrow q \vee p$ and $M \models p \vee q \Rightarrow q \vee p$.

(126) $M, v \models p \Rightarrow p \vee p$ and $M \models p \Rightarrow p \vee p$.

(127) $M, v \models (p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r))$ and $M \models (p \Rightarrow r) \Rightarrow ((q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r))$.

(128) $M, v \models (p \Rightarrow r) \wedge (q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)$ and $M \models (p \Rightarrow r) \wedge (q \Rightarrow r) \Rightarrow (p \vee q \Rightarrow r)$.

(129) $M, v \models (p \Rightarrow \neg q) \Rightarrow (q \Rightarrow \neg p)$ and $M \models (p \Rightarrow \neg q) \Rightarrow (q \Rightarrow \neg p)$.

(130) $M, v \models \neg p \Rightarrow (p \Rightarrow q)$ and $M \models \neg p \Rightarrow (p \Rightarrow q)$.

(131) $M, v \models (p \Rightarrow q) \wedge (p \Rightarrow \neg q) \Rightarrow \neg p$ and $M \models (p \Rightarrow q) \wedge (p \Rightarrow \neg q) \Rightarrow \neg p$.

(133)[4] If $M \models p \Rightarrow q$ and $M \models p$, then $M \models q$.

(134) $M, v \models \neg(p \wedge q) \Rightarrow \neg p \vee \neg q$ and $M \models \neg(p \wedge q) \Rightarrow \neg p \vee \neg q$.

(135) $M, v \models \neg p \vee \neg q \Rightarrow \neg(p \wedge q)$ and $M \models \neg p \vee \neg q \Rightarrow \neg(p \wedge q)$.

(136) $M, v \models \neg(p \vee q) \Rightarrow \neg p \wedge \neg q$ and $M \models \neg(p \vee q) \Rightarrow \neg p \wedge \neg q$.

(137) $M, v \models \neg p \wedge \neg q \Rightarrow \neg(p \vee q)$ and $M \models \neg p \wedge \neg q \Rightarrow \neg(p \vee q)$.

(138) $M \models \forall_x H \Rightarrow H$.

(139) $M \models H \Rightarrow \exists_x H$.

(140) If $x \notin \mathrm{Free}\, H_1$, then $M \models \forall_x (H_1 \Rightarrow H_2) \Rightarrow (H_1 \Rightarrow \forall_x H_2)$.

(141) If $x \notin \mathrm{Free}\, H_1$ and $M \models H_1 \Rightarrow H_2$, then $M \models H_1 \Rightarrow \forall_x H_2$.

(142) If $x \notin \mathrm{Free}\, H_2$, then $M \models \forall_x (H_1 \Rightarrow H_2) \Rightarrow (\exists_x H_1 \Rightarrow H_2)$.

(143) If $x \notin \mathrm{Free}\, H_2$ and $M \models H_1 \Rightarrow H_2$, then $M \models \exists_x H_1 \Rightarrow H_2$.

(144) If $M \models H_1 \Rightarrow \forall_x H_2$, then $M \models H_1 \Rightarrow H_2$.

(145) If $M \models \exists_x H_1 \Rightarrow H_2$, then $M \models H_1 \Rightarrow H_2$.

(146) $\mathrm{WFF} \subseteq 2^{[:\mathbb{N}, \mathbb{N}:]}$.

Let us consider H. The functor Var_H yields a set and is defined as follows:

(Def. 3) $\mathrm{Var}_H = \mathrm{rng}\, H \setminus \{0, 1, 2, 3, 4\}$.

The following propositions are true:

(148)[5] $x \neq 0$ and $x \neq 1$ and $x \neq 2$ and $x \neq 3$ and $x \neq 4$.

(149) $x \notin \{0, 1, 2, 3, 4\}$.

(150) If $a \in \mathrm{Var}_H$, then $a \neq 0$ and $a \neq 1$ and $a \neq 2$ and $a \neq 3$ and $a \neq 4$.

[4] The proposition (132) has been removed.
[5] The proposition (147) has been removed.

(151) $\mathrm{Var}_{x=y} = \{x,y\}$.

(152) $\mathrm{Var}_{x\varepsilon y} = \{x,y\}$.

(153) $\mathrm{Var}_{\neg H} = \mathrm{Var}_H$.

(154) $\mathrm{Var}_{H_1 \wedge H_2} = \mathrm{Var}_{(H_1)} \cup \mathrm{Var}_{(H_2)}$.

(155) $\mathrm{Var}_{\forall_x H} = \mathrm{Var}_H \cup \{x\}$.

(156) $\mathrm{Var}_{H_1 \vee H_2} = \mathrm{Var}_{(H_1)} \cup \mathrm{Var}_{(H_2)}$.

(157) $\mathrm{Var}_{H_1 \Rightarrow H_2} = \mathrm{Var}_{(H_1)} \cup \mathrm{Var}_{(H_2)}$.

(158) $\mathrm{Var}_{H_1 \Leftrightarrow H_2} = \mathrm{Var}_{(H_1)} \cup \mathrm{Var}_{(H_2)}$.

(159) $\mathrm{Var}_{\exists_x H} = \mathrm{Var}_H \cup \{x\}$.

(160) $\mathrm{Var}_{\forall_{x,y} H} = \mathrm{Var}_H \cup \{x,y\}$.

(161) $\mathrm{Var}_{\exists_{x,y} H} = \mathrm{Var}_H \cup \{x,y\}$.

(162) $\mathrm{Var}_{\forall_{x,y,z} H} = \mathrm{Var}_H \cup \{x,y,z\}$.

(163) $\mathrm{Var}_{\exists_{x,y,z} H} = \mathrm{Var}_H \cup \{x,y,z\}$.

(164) $\mathrm{Free}\,H \subseteq \mathrm{Var}_H$.

Let us consider H. Then Var_H is a non empty subset of VAR.

Let us consider H, x, y. The functor $H(\frac{x}{y})$ yields a function and is defined by:

(Def. 4) $\mathrm{dom}(H(\frac{x}{y})) = \mathrm{dom}\,H$ and for every a such that $a \in \mathrm{dom}\,H$ holds if $H(a) = x$, then $H(\frac{x}{y})(a) = y$ and if $H(a) \neq x$, then $H(\frac{x}{y})(a) = H(a)$.

We now state several propositions:

(166)[6] $x_1{=}x_2(\frac{y_1}{y_2}) = z_1{=}z_2$ if and only if one of the following conditions is satisfied:

(i) $x_1 \neq y_1$ and $x_2 \neq y_1$ and $z_1 = x_1$ and $z_2 = x_2$, or

(ii) $x_1 = y_1$ and $x_2 \neq y_1$ and $z_1 = y_2$ and $z_2 = x_2$, or

(iii) $x_1 \neq y_1$ and $x_2 = y_1$ and $z_1 = x_1$ and $z_2 = y_2$, or

(iv) $x_1 = y_1$ and $x_2 = y_1$ and $z_1 = y_2$ and $z_2 = y_2$.

(167) There exist z_1, z_2 such that $x_1{=}x_2(\frac{y_1}{y_2}) = z_1{=}z_2$.

(168) $x_1\varepsilon x_2(\frac{y_1}{y_2}) = z_1\varepsilon z_2$ if and only if one of the following conditions is satisfied:

(i) $x_1 \neq y_1$ and $x_2 \neq y_1$ and $z_1 = x_1$ and $z_2 = x_2$, or

(ii) $x_1 = y_1$ and $x_2 \neq y_1$ and $z_1 = y_2$ and $z_2 = x_2$, or

(iii) $x_1 \neq y_1$ and $x_2 = y_1$ and $z_1 = x_1$ and $z_2 = y_2$, or

(iv) $x_1 = y_1$ and $x_2 = y_1$ and $z_1 = y_2$ and $z_2 = y_2$.

(169) There exist z_1, z_2 such that $x_1\varepsilon x_2(\frac{y_1}{y_2}) = z_1\varepsilon z_2$.

(170) $\neg F = (\neg H)(\frac{x}{y})$ iff $F = H(\frac{x}{y})$.

(171) $H(\frac{x}{y}) \in \mathrm{WFF}$.

Let us consider H, x, y. Then $H(\frac{x}{y})$ is a ZF-formula.

One can prove the following propositions:

[6] The proposition (165) has been removed.

(172) $G_1 \wedge G_2 = (H_1 \wedge H_2)(\frac{x}{y})$ iff $G_1 = H_1(\frac{x}{y})$ and $G_2 = H_2(\frac{x}{y})$.

(173) If $z \neq x$, then $\forall_z G = (\forall_z H)(\frac{x}{y})$ iff $G = H(\frac{x}{y})$.

(174) $\forall_y G = (\forall_x H)(\frac{x}{y})$ iff $G = H(\frac{x}{y})$.

(175) $G_1 \vee G_2 = (H_1 \vee H_2)(\frac{x}{y})$ iff $G_1 = H_1(\frac{x}{y})$ and $G_2 = H_2(\frac{x}{y})$.

(176) $G_1 \Rightarrow G_2 = (H_1 \Rightarrow H_2)(\frac{x}{y})$ iff $G_1 = H_1(\frac{x}{y})$ and $G_2 = H_2(\frac{x}{y})$.

(177) $G_1 \Leftrightarrow G_2 = (H_1 \Leftrightarrow H_2)(\frac{x}{y})$ iff $G_1 = H_1(\frac{x}{y})$ and $G_2 = H_2(\frac{x}{y})$.

(178) If $z \neq x$, then $\exists_z G = (\exists_z H)(\frac{x}{y})$ iff $G = H(\frac{x}{y})$.

(179) $\exists_y G = (\exists_x H)(\frac{x}{y})$ iff $G = H(\frac{x}{y})$.

(180) H is an equality iff $H(\frac{x}{y})$ is an equality.

(181) H is a membership iff $H(\frac{x}{y})$ is a membership.

(182) H is negative iff $H(\frac{x}{y})$ is negative.

(183) H is conjunctive iff $H(\frac{x}{y})$ is conjunctive.

(184) H is universal iff $H(\frac{x}{y})$ is universal.

(185) If H is negative, then $\mathrm{Arg}(H(\frac{x}{y})) = \mathrm{Arg}(H)(\frac{x}{y})$.

(186) If H is conjunctive, then $\mathrm{LeftArg}(H(\frac{x}{y})) = \mathrm{LeftArg}(H)(\frac{x}{y})$ and $\mathrm{RightArg}(H(\frac{x}{y})) = \mathrm{RightArg}(H)(\frac{x}{y})$.

(187) If H is universal, then $\mathrm{Scope}(H(\frac{x}{y})) = \mathrm{Scope}(H)(\frac{x}{y})$ and if $\mathrm{Bound}(H) = x$, then $\mathrm{Bound}(H(\frac{x}{y})) = y$ and if $\mathrm{Bound}(H) \neq x$, then $\mathrm{Bound}(H(\frac{x}{y})) = \mathrm{Bound}(H)$.

(188) H is disjunctive iff $H(\frac{x}{y})$ is disjunctive.

(189) H is conditional iff $H(\frac{x}{y})$ is conditional.

(190) If H is biconditional, then $H(\frac{x}{y})$ is biconditional.

(191) H is existential iff $H(\frac{x}{y})$ is existential.

(192) If H is disjunctive, then $\mathrm{LeftArg}(H(\frac{x}{y})) = \mathrm{LeftArg}(H)(\frac{x}{y})$ and $\mathrm{RightArg}(H(\frac{x}{y})) = \mathrm{RightArg}(H)(\frac{x}{y})$.

(193) If H is conditional, then $\mathrm{Antecedent}(H(\frac{x}{y})) = \mathrm{Antecedent}(H)(\frac{x}{y})$ and $\mathrm{Consequent}(H(\frac{x}{y})) = \mathrm{Consequent}(H)(\frac{x}{y})$.

(194) If H is biconditional, then $\mathrm{LeftSide}(H(\frac{x}{y})) = \mathrm{LeftSide}(H)(\frac{x}{y})$ and $\mathrm{RightSide}(H(\frac{x}{y})) = \mathrm{RightSide}(H)(\frac{x}{y})$.

(195) If H is existential, then $\mathrm{Scope}(H(\frac{x}{y})) = \mathrm{Scope}(H)(\frac{x}{y})$ and if $\mathrm{Bound}(H) = x$, then $\mathrm{Bound}(H(\frac{x}{y})) = y$ and if $\mathrm{Bound}(H) \neq x$, then $\mathrm{Bound}(H(\frac{x}{y})) = \mathrm{Bound}(H)$.

(196) If $x \notin \mathrm{Var}_H$, then $H(\frac{x}{y}) = H$.

(197) $H(\frac{x}{x}) = H$.

(198) If $x \neq y$, then $x \notin \mathrm{Var}_{H(\frac{x}{y})}$.

(199) If $x \in \mathrm{Var}_H$, then $y \in \mathrm{Var}_{H(\frac{x}{y})}$.

(200) If $x \neq y$, then $H(\frac{x}{y})(\frac{x}{z}) = H(\frac{x}{y})$.

(201) $\mathrm{Var}_{H(\frac{x}{y})} \subseteq (\mathrm{Var}_H \setminus \{x\}) \cup \{y\}$.

REFERENCES

[1] Grzegorz Bancerek. A model of ZF set theory language. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zf_lang.html.

[2] Grzegorz Bancerek. Models and satisfiability. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zf_model.html.

[3] Grzegorz Bancerek. Properties of ZF models. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmodel1.html.

[4] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finseq_1.html.

[5] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[6] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[7] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[8] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/finset_1.html.

[9] Andrzej Trybulec. Enumerated sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/enumset1.html.

[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[11] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. http://mizar.org/JFM/Addenda/numbers.html.

[12] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[13] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

Received August 10, 1990

Published January 2, 2004