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Inst. of Computer Science, Univ. of Białystok

Baire Spaces, Sober Spaces[1]

Andrzej Trybulec
Warsaw University
Białystok

Summary. In the article concepts and facts necessary to continue formalization of theory of continuous lattices according to [11] are introduced.

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The articles [16], [8], [18], [19], [7], [15], [13], [2], [1], [3], [5], [10], [17], [14], [20], [12], [6], [9], and [4] provide the notation and terminology for this paper.

1. Preliminaries

The following two propositions are true:

(1) For all sets X, A, B such that $A \in \mathrm{Fin}X$ and $B \subseteq A$ holds $B \in \mathrm{Fin}X$.

(2) For every set X and for every family F of subsets of X such that $F \subseteq \mathrm{Fin}X$ holds $\bigcap F \in \mathrm{Fin}X$.

Let X be a non empty set. Let us observe that X is trivial if and only if:

(Def. 1) For all elements x, y of X holds $x = y$.

2. Families of complements

Next we state several propositions:

(3) For every set X and for every family F of subsets of X and for every subset P of X holds $P^{\rm c} \in F^{\rm c}$ iff $P \in F$.

(4) For every set X and for every family F of subsets of X holds $F \approx F^{\rm c}$.

(5) For all sets X, Y such that $X \approx Y$ and X is countable holds Y is countable.

(6) For every set X and for every family F of subsets of X holds $(F^{\rm c})^{\rm c} = F$.

(7) For every set X and for every family F of subsets of X and for every subset P of X holds $P^{\rm c} \in F^{\rm c}$ iff $P \in F$.

(8) For every set X and for all families F, G of subsets of X such that $F^{\rm c} \subseteq G^{\rm c}$ holds $F \subseteq G$.

(9) For every set X and for all families F, G of subsets of X holds $F^{\rm c} \subseteq G$ iff $F \subseteq G^{\rm c}$.

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(10) For every set X and for all families F, G of subsets of X such that $F^{\rm c} = G^{\rm c}$ holds $F = G$.

Let X be a set and let F, G be subsets of 2^X. Then $F \cup G$ is a family of subsets of X.
Next we state two propositions:

(11) For every set X and for all families F, G of subsets of X holds $(F \cup G)^{\rm c} = F^{\rm c} \cup G^{\rm c}$.

(12) For every set X and for every family F of subsets of X such that $F = \{X\}$ holds $F^{\rm c} = \{\emptyset\}$.

Let X be a set and let F be an empty family of subsets of X. One can check that $F^{\rm c}$ is empty.
We now state four propositions:

(13) Let X be a 1-sorted structure, F be a family of subsets of X, and P be a subset of X. Then $P \in F^{\rm c}$ if and only if $P^{\rm c} \in F$.

(14) Let X be a 1-sorted structure, F be a family of subsets of X, and P be a subset of X. Then $P^{\rm c} \in F^{\rm c}$ if and only if $P \in F$.

(15) For every 1-sorted structure X and for every family F of subsets of X holds $\mathrm{Intersect}(F^{\rm c}) = (\bigcup F)^{\rm c}$.

(16) For every 1-sorted structure X and for every family F of subsets of X holds $\bigcup(F^{\rm c}) = (\mathrm{Intersect}(F))^{\rm c}$.

3. Topological preliminaries

One can prove the following four propositions:

(17) Let T be a non empty topological space and A, B be subsets of T. Suppose $B \subseteq A$ and A is closed and for every subset C of T such that $B \subseteq C$ and C is closed holds $A \subseteq C$. Then $A = \overline{B}$.

(18) Let T be a topological structure, B be a basis of T, and V be a subset of T. If V is open, then $V = \bigcup\{G; G$ ranges over subsets of T: $G \in B \wedge G \subseteq V\}$.

(19) Let T be a topological structure, B be a basis of T, and S be a subset of T. If $S \in B$, then S is open.

(20) Let T be a non empty topological space, B be a basis of T, and V be a subset of T. Then $\mathrm{Int}\,V = \bigcup\{G; G$ ranges over subsets of T: $G \in B \wedge G \subseteq V\}$.

4. Baire Spaces

Let T be a non empty topological structure and let x be a point of T. A family of subsets of T is said to be a basis of x if it satisfies the conditions (Def. 2).

(Def. 2)(i) It $\subseteq$ the topology of T,

(ii) $x \in \mathrm{Intersect}(\mathrm{it})$, and

(iii) for every subset S of T such that S is open and $x \in S$ there exists a subset V of T such that $V \in$ it and $V \subseteq S$.

One can prove the following three propositions:

(21) Let T be a non empty topological structure, x be a point of T, B be a basis of x, and V be a subset of T. If $V \in B$, then V is open and $x \in V$.

(22) Let T be a non empty topological structure, x be a point of T, B be a basis of x, and V be a subset of T. If $x \in V$ and V is open, then there exists a subset W of T such that $W \in B$ and $W \subseteq V$.

(23) Let T be a non empty topological structure and P be a family of subsets of T. Suppose $P \subseteq$ the topology of T and for every point x of T there exists a basis B of x such that $B \subseteq P$. Then P is a basis of T.

Let T be a non empty topological space. We say that T is Baire if and only if the condition (Def. 3) is satisfied.

(Def. 3) Let F be a family of subsets of T. Suppose F is countable and for every subset S of T such that $S \in F$ holds S is everywhere dense. Then there exists a subset I of T such that $I = \mathrm{Intersect}(F)$ and I is dense.

One can prove the following proposition

(24) Let T be a non empty topological space. Then T is Baire if and only if for every family F of subsets of T such that F is countable and for every subset S of T such that $S \in F$ holds S is nowhere dense holds $\bigcup F$ is boundary.

5. Sober Spaces

Let T be a topological structure and let S be a subset of T. We say that S is irreducible if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) S is non empty and closed, and

(ii) for all subsets S_1, S_2 of T such that S_1 is closed and S_2 is closed and $S = S_1 \cup S_2$ holds $S_1 = S$ or $S_2 = S$.

Let T be a topological structure. Note that every subset of T which is irreducible is also non empty.

Let T be a non empty topological space, let S be a subset of T, and let p be a point of T. We say that p is dense point of S if and only if:

(Def. 5) $p \in S$ and $S \subseteq \overline{\{p\}}$.

The following propositions are true:

(25) Let T be a non empty topological space and S be a subset of T. Suppose S is closed. Let p be a point of T. If p is dense point of S, then $S = \overline{\{p\}}$.

(26) For every non empty topological space T and for every point p of T holds $\overline{\{p\}}$ is irreducible.

Let T be a non empty topological space. Observe that there exists a subset of T which is irreducible.

Let T be a non empty topological space. We say that T is sober if and only if the condition (Def. 6) is satisfied.

(Def. 6) Let S be an irreducible subset of T. Then there exists a point p of T such that p is dense point of S and for every point q of T such that q is dense point of S holds $p = q$.

Next we state four propositions:

(27) For every non empty topological space T holds every point p of T is dense point of $\overline{\{p\}}$.

(28) For every non empty topological space T holds every point p of T is dense point of $\{p\}$.

(29) Let T be a non empty topological space and G, F be subsets of T. If G is open and F is closed, then $F \setminus G$ is closed.

(30) For every Hausdorff non empty topological space T holds every irreducible subset of T is trivial.

Let T be a Hausdorff non empty topological space. One can check that every subset of T which is irreducible is also trivial.

The following proposition is true

(31) Every Hausdorff non empty topological space is sober.

Let us note that every non empty topological space which is Hausdorff is also sober.

Let us observe that there exists a non empty topological space which is sober.

We now state two propositions:

(32) Let T be a non empty topological space. Then T is T_0 if and only if for all points p, q of T such that $\overline{\{p\}} = \overline{\{q\}}$ holds $p = q$.

(33) Every sober non empty topological space is T_0.

Let us observe that every non empty topological space which is sober is also T_0.

Let X be a set. The functor $\mathrm{CofinTop}\,X$ yielding a strict topological structure is defined as follows:

(Def. 7) The carrier of $\mathrm{CofinTop}\,X = X$ and (the topology of $\mathrm{CofinTop}\,X)^{\mathrm{c}} = \{X\} \cup \mathrm{Fin}\,X$.

Let X be a non empty set. One can verify that $\mathrm{CofinTop}\,X$ is non empty.

Let X be a set. One can verify that $\mathrm{CofinTop}\,X$ is topological space-like.

Next we state two propositions:

(34) For every non empty set X and for every subset P of $\mathrm{CofinTop}\,X$ holds P is closed iff $P = X$ or P is finite.

(35) For every non empty topological space T such that T is a T_1 space and for every point p of T holds $\overline{\{p\}} = \{p\}$.

Let X be a non empty set. Note that $\mathrm{CofinTop}\,X$ is T_1.

Let X be an infinite set. Note that $\mathrm{CofinTop}\,X$ is non sober.

Let us observe that there exists a non empty topological space which is T_1 and non sober.

6. More on regular spaces

One can prove the following propositions:

(36) Let T be a non empty topological space. Then T is a T_3 space if and only if for every point p of T and for every subset P of T such that $p \in \mathrm{Int}\,P$ there exists a subset Q of T such that Q is closed and $Q \subseteq P$ and $p \in \mathrm{Int}\,Q$.

(37) Let T be a non empty topological space. Suppose T is a T_3 space. Then T is locally-compact if and only if for every point x of T there exists a subset Y of T such that $x \in \mathrm{Int}\,Y$ and Y is compact.

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