JOURNAL OF FORMALIZED MATHEMATICS
Volume 8, Released 1996, Published 2003
Inst. of Computer Science, Univ. of Białystok

Boolean Posets, Posets under Inclusion and Products of Relational Structures[1]

Adam Grabowski
Warsaw University
Białystok

Robert Milewski
Warsaw University
Białystok

Summary. In the paper some notions useful in formalization of [11] are introduced, e.g. the definition of the poset of subsets of a set with inclusion as an ordering relation. Using the theory of many sorted sets authors formulate the definition of product of relational structures.

MML Identifier: YELLOW_1.

WWW: http://mizar.org/JFM/Vol8/yellow_1.html

The articles [17], [9], [20], [21], [23], [22], [15], [5], [6], [10], [1], [8], [7], [19], [24], [12], [3], [16], [14], [18], [2], [13], and [4] provide the notation and terminology for this paper.

1. Boolean Posets and Posets under Inclusion

In this paper X denotes a set.

Let L be a lattice. Note that Poset(L) has l.u.b.'s and g.l.b.'s.

Let L be an upper-bounded lattice. One can verify that Poset(L) is upper-bounded.

Let L be a lower-bounded lattice. One can verify that Poset(L) is lower-bounded.

Let L be a complete lattice. Note that Poset(L) is complete.

Let X be a set. Then ${}^{\subseteq}X$ is an order in X.

Let X be a set. The functor $\langle X, \subseteq \rangle$ yielding a strict relational structure is defined by:

(Def. 1) $\langle X, \subseteq \rangle = \langle X, {}^{\subseteq}X \rangle$.

Let X be a set. Observe that $\langle X, \subseteq \rangle$ is reflexive, antisymmetric, and transitive.

Let X be a non empty set. Observe that $\langle X, \subseteq \rangle$ is non empty.

The following proposition is true

(1) The carrier of $(\langle X, \subseteq \rangle) = X$ and the internal relation of $(\langle X, \subseteq \rangle) = {}^{\subseteq}X$.

Let X be a set. The functor $2^X_{\subseteq}$ yields a strict relational structure and is defined as follows:

(Def. 2) $2^X_{\subseteq} =$ Poset(the lattice of subsets of X).

Let X be a set. One can check that $2^X_{\subseteq}$ is non empty, reflexive, antisymmetric, and transitive.

Let X be a set. Note that $2^X_{\subseteq}$ is complete.

The following propositions are true:

(2) For all elements x, y of $2^X_{\subseteq}$ holds $x \leq y$ iff $x \subseteq y$.

[1]This work was partially supported by Office of Naval Research Grant N00014-95-1-1336.

(3) For every non empty set X and for all elements x, y of $\langle X, \subseteq \rangle$ holds $x \leq y$ iff $x \subseteq y$.

(4) $2^X_\subseteq = \langle 2^X, \subseteq \rangle$.

(5) For every subset Y of 2^X holds $\langle Y, \subseteq \rangle$ is a full relational substructure of $2^X_\subseteq$.

(6) For every non empty set X such that $\langle X, \subseteq \rangle$ has l.u.b.'s and for all elements x, y of $\langle X, \subseteq \rangle$ holds $x \cup y \subseteq x \sqcup y$.

(7) For every non empty set X such that $\langle X, \subseteq \rangle$ has g.l.b.'s and for all elements x, y of $\langle X, \subseteq \rangle$ holds $x \sqcap y \subseteq x \cap y$.

(8) For every non empty set X and for all elements x, y of $\langle X, \subseteq \rangle$ such that $x \cup y \in X$ holds $x \sqcup y = x \cup y$.

(9) For every non empty set X and for all elements x, y of $\langle X, \subseteq \rangle$ such that $x \cap y \in X$ holds $x \sqcap y = x \cap y$.

(10) Let L be a relational structure. Suppose that for all elements x, y of L holds $x \leq y$ iff $x \subseteq y$. Then the internal relation of $L = \subseteq_{\text{the carrier of } L}$.

(11) For every non empty set X such that for all sets x, y such that $x \in X$ and $y \in X$ holds $x \cup y \in X$ holds $\langle X, \subseteq \rangle$ has l.u.b.'s.

(12) For every non empty set X such that for all sets x, y such that $x \in X$ and $y \in X$ holds $x \cap y \in X$ holds $\langle X, \subseteq \rangle$ has g.l.b.'s.

(13) For every non empty set X such that $\emptyset \in X$ holds $\perp_{\langle X, \subseteq \rangle} = \emptyset$.

(14) For every non empty set X such that $\bigcup X \in X$ holds $\top_{\langle X, \subseteq \rangle} = \bigcup X$.

(15) For every non empty set X such that $\langle X, \subseteq \rangle$ is upper-bounded holds $\bigcup X \in X$.

(16) For every non empty set X such that $\langle X, \subseteq \rangle$ is lower-bounded holds $\bigcap X \in X$.

(17) For all elements x, y of $2^X_\subseteq$ holds $x \sqcup y = x \cup y$ and $x \sqcap y = x \cap y$.

(18) $\perp_{2^X_\subseteq} = \emptyset$.

(19) $\top_{2^X_\subseteq} = X$.

(20) For every non empty subset Y of $2^X_\subseteq$ holds $\inf Y = \bigcap Y$.

(21) For every subset Y of $2^X_\subseteq$ holds $\sup Y = \bigcup Y$.

(22) For every non empty topological space T and for every subset X of $\langle$the topology of T, $\subseteq\rangle$ holds $\sup X = \bigcup X$.

(23) For every non empty topological space T holds $\perp_{\langle \text{the topology of } T, \subseteq \rangle} = \emptyset$.

(24) For every non empty topological space T holds $\top_{\langle \text{the topology of } T, \subseteq \rangle} =$ the carrier of T.

Let T be a non empty topological space. Note that $\langle$the topology of T, $\subseteq\rangle$ is complete and non trivial.

The following proposition is true

(25) Let T be a topological space and F be a family of subsets of T. Then F is open if and only if F is a subset of $\langle$the topology of T, $\subseteq\rangle$.

2. Products of Relational Structures

Let R be a binary relation. We say that R is relational structure yielding if and only if:

(Def. 3) For every set v such that $v \in \operatorname{rng} R$ holds v is a relational structure.

Let us observe that every function which is relational structure yielding is also 1-sorted yielding.

Let I be a set. Note that there exists a many sorted set indexed by I which is relational structure yielding.

Let J be a non empty set, let A be a relational structure yielding many sorted set indexed by J, and let j be an element of J. Then $A(j)$ is a relational structure.

Let I be a set and let J be a relational structure yielding many sorted set indexed by I. The functor $\prod J$ yields a strict relational structure and is defined by the conditions (Def. 4).

(Def. 4)(i) The carrier of $\prod J = \prod$(the support of J), and

(ii) for all elements x, y of $\prod J$ such that $x \in \prod$(the support of J) holds $x \leq y$ iff there exist functions f, g such that $f = x$ and $g = y$ and for every set i such that $i \in I$ there exists a relational structure R and there exist elements x_1, y_1 of R such that $R = J(i)$ and $x_1 = f(i)$ and $y_1 = g(i)$ and $x_1 \leq y_1$.

Let X be a set and let L be a relational structure. Note that $X \longmapsto L$ is relational structure yielding.

Let I be a set and let T be a relational structure. The functor T^I yielding a strict relational structure is defined by:

(Def. 5) $T^I = \prod(I \longmapsto T)$.

One can prove the following propositions:

(26) For every relational structure yielding many sorted set J indexed by $\emptyset$ holds $\prod J = \langle\{\emptyset\}, \mathrm{id}_{\{\emptyset\}}\rangle$.

(27) For every relational structure Y holds $Y^{\emptyset} = \langle\{\emptyset\}, \mathrm{id}_{\{\emptyset\}}\rangle$.

(28) For every set X and for every relational structure Y holds (the carrier of $Y)^X =$ the carrier of Y^X.

Let X be a set and let Y be a non empty relational structure. Observe that Y^X is non empty.

Let X be a set and let Y be a reflexive non empty relational structure. Observe that Y^X is reflexive.

Let Y be a non empty relational structure. Note that $Y^{\emptyset}$ is trivial.

Let Y be a non empty reflexive relational structure. Observe that $Y^{\emptyset}$ is antisymmetric and has g.l.b.'s and l.u.b.'s.

Let X be a set and let Y be a transitive non empty relational structure. One can check that Y^X is transitive.

Let X be a set and let Y be an antisymmetric non empty relational structure. One can verify that Y^X is antisymmetric.

Let X be a non empty set and let Y be a non empty antisymmetric relational structure with g.l.b.'s. One can check that Y^X has g.l.b.'s.

Let X be a non empty set and let Y be a non empty antisymmetric relational structure with l.u.b.'s. Observe that Y^X has l.u.b.'s.

Let S, T be relational structures. The functor $\mathrm{MonMaps}(S,T)$ yields a strict full relational substructure of $T^{\text{the carrier of } S}$ and is defined by the condition (Def. 6).

(Def. 6) Let f be a map from S into T. Then $f \in$ the carrier of $\mathrm{MonMaps}(S,T)$ if and only if $f \in$ (the carrier of $T)^{\text{the carrier of } S}$ and f is monotone.

ACKNOWLEDGMENTS

The authors would like to express their gratitude to Professor Andrzej Trybulec for his help in formulating mizared definition of the product.

REFERENCES

[1] Grzegorz Bancerek. Zermelo theorem and axiom of choice. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/wellord2.html.

[2] Grzegorz Bancerek. König's theorem. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/card_3.html.

[3] Grzegorz Bancerek. Complete lattices. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/lattice3.html.

[4] Grzegorz Bancerek. Bounds in posets and relational substructures. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/yellow_0.html.

[5] Józef Białas. Group and field definitions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/realset1.html.

[6] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[7] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[8] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[9] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[10] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/tops_2.html.

[11] G. Gierz, K.H. Hofmann, K. Keimel, J.D. Lawson, M. Mislove, and D.S. Scott. *A Compendium of Continuous Lattices*. Springer-Verlag, Berlin, Heidelberg, New York, 1980.

[12] Adam Grabowski. On the category of posets. *Journal of Formalized Mathematics*, 8, 1996. http://mizar.org/JFM/Vol8/orders_3.html.

[13] Beata Madras. Product of family of universal algebras. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pralg_1.html.

[14] Yatsuka Nakamura, Piotr Rudnicki, Andrzej Trybulec, and Pauline N. Kawamoto. Preliminaries to circuits, I. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/pre_circ.html.

[15] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/setfam_1.html.

[16] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/pre_topc.html.

[17] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[18] Andrzej Trybulec. Many-sorted sets. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/pboole.html.

[19] Wojciech A. Trybulec. Partially ordered sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/orders_1.html.

[20] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[21] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_1.html.

[22] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relset_1.html.

[23] Edmund Woronowicz and Anna Zalewska. Properties of binary relations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/relat_2.html.

[24] Stanisław Żukowski. Introduction to lattice theory. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/lattices.html`.

Received September 20, 1996

Published January 2, 2004