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Bounds in Posets and Relational Substructures[1]

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Summary. Notation and facts necessary to start with the formalization of continuous lattices according to [4] are introduced.

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The articles [6], [2], [8], [10], [9], [3], [5], [11], [7], and [1] provide the notation and terminology for this paper.

1. Reexamination of Poset Concepts

The scheme *RelStrEx* deals with a non empty set $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

There exists a non empty strict relational structure L such that the carrier of $L = \mathcal{A}$ and for all elements a, b of L holds $a \leq b$ iff $\mathcal{P}[a,b]$

for all values of the parameters.

Let A be a non empty relational structure. Let us observe that A is reflexive if and only if:

(Def. 1) For every element x of A holds $x \leq x$.

Let A be a relational structure. Let us observe that A is transitive if and only if:

(Def. 2) For all elements x, y, z of A such that $x \leq y$ and $y \leq z$ holds $x \leq z$.

Let us observe that A is antisymmetric if and only if:

(Def. 3) For all elements x, y of A such that $x \leq y$ and $y \leq x$ holds $x = y$.

Let us note that every non empty relational structure which is complete has also l.u.b.'s and g.l.b.'s and every non empty reflexive relational structure which is trivial is also complete, transitive, and antisymmetric.

Let x be a set and let R be a binary relation on $\{x\}$. One can check that $\langle \{x\}, R \rangle$ is trivial.

Let us observe that there exists a relational structure which is strict, trivial, non empty, and reflexive.

Next we state a number of propositions:

(1) Let P_1, P_2 be relational structures. Suppose the relational structure of $P_1 =$ the relational structure of P_2. Let a_1, b_1 be elements of P_1 and a_2, b_2 be elements of P_2 such that $a_1 = a_2$ and $b_1 = b_2$. Then

(i) if $a_1 \leq b_1$, then $a_2 \leq b_2$, and

(ii) if $a_1 < b_1$, then $a_2 < b_2$.

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(2) Let P_1, P_2 be relational structures. Suppose the relational structure of P_1 = the relational structure of P_2. Let X be a set, a_1 be an element of P_1, and a_2 be an element of P_2 such that $a_1 = a_2$. Then

(i) if $X \le a_1$, then $X \le a_2$, and

(ii) if $X \ge a_1$, then $X \ge a_2$.

(3) Let P_1, P_2 be relational structures. Suppose the relational structure of P_1 = the relational structure of P_2 and P_1 is complete. Then P_2 is complete.

(4) Let L be a transitive relational structure and x, y be elements of L. Suppose $x \le y$. Let X be a set. Then

(i) if $y \le X$, then $x \le X$, and

(ii) if $x \ge X$, then $y \ge X$.

(5) Let L be a non empty relational structure, X be a set, and x be an element of L. Then

(i) $x \ge X$ iff $x \ge X \cap$ the carrier of L, and

(ii) $x \le X$ iff $x \le X \cap$ the carrier of L.

(6) For every relational structure L and for every element a of L holds $\emptyset \le a$ and $\emptyset \ge a$.

(7) Let L be a relational structure and a, b be elements of L. Then

(i) $a \le \{b\}$ iff $a \le b$, and

(ii) $a \ge \{b\}$ iff $b \le a$.

(8) Let L be a relational structure and a, b, c be elements of L. Then

(i) $a \le \{b, c\}$ iff $a \le b$ and $a \le c$, and

(ii) $a \ge \{b, c\}$ iff $b \le a$ and $c \le a$.

(9) Let L be a relational structure and X, Y be sets. Suppose $X \subseteq Y$. Let x be an element of L. Then

(i) if $x \le Y$, then $x \le X$, and

(ii) if $x \ge Y$, then $x \ge X$.

(10) Let L be a relational structure, X, Y be sets, and x be an element of L. Then

(i) if $x \le X$ and $x \le Y$, then $x \le X \cup Y$, and

(ii) if $x \ge X$ and $x \ge Y$, then $x \ge X \cup Y$.

(11) Let L be a transitive relational structure, X be a set, and x, y be elements of L. If $X \le x$ and $x \le y$, then $X \le y$.

(12) Let L be a transitive relational structure, X be a set, and x, y be elements of L. If $X \ge x$ and $x \ge y$, then $X \ge y$.

Let L be a non empty relational structure. Note that Ω_L is non empty.

2. Least upper and greatest lower bounds

Let L be a relational structure. We say that L is lower-bounded if and only if:

(Def. 4) There exists an element x of L such that $x \le$ the carrier of L.

We say that L is upper-bounded if and only if:

(Def. 5) There exists an element x of L such that $x \ge$ the carrier of L.

Let L be a relational structure. We say that L is bounded if and only if:

(Def. 6) L is lower-bounded and upper-bounded.

Next we state the proposition

(13) Let P_1, P_2 be relational structures such that the relational structure of $P_1 =$ the relational structure of P_2. Then

(i) if P_1 is lower-bounded, then P_2 is lower-bounded, and

(ii) if P_1 is upper-bounded, then P_2 is upper-bounded.

One can verify the following observations:

* every non empty relational structure which is complete is also bounded,
* every relational structure which is bounded is also lower-bounded and upper-bounded, and
* every relational structure which is lower-bounded and upper-bounded is also bounded.

Let us mention that there exists a non empty poset which is complete.

Let L be a relational structure and let X be a set. We say that sup X exists in L if and only if the condition (Def. 7) is satisfied.

(Def. 7) There exists an element a of L such that

(i) $X \leq a$,

(ii) for every element b of L such that $X \leq b$ holds $b \geq a$, and

(iii) for every element c of L such that $X \leq c$ and for every element b of L such that $X \leq b$ holds $b \geq c$ holds $c = a$.

We say that inf X exists in L if and only if the condition (Def. 8) is satisfied.

(Def. 8) There exists an element a of L such that

(i) $X \geq a$,

(ii) for every element b of L such that $X \geq b$ holds $b \leq a$, and

(iii) for every element c of L such that $X \geq c$ and for every element b of L such that $X \geq b$ holds $b \leq c$ holds $c = a$.

Next we state a number of propositions:

(14) Let L_1, L_2 be relational structures. Suppose the relational structure of $L_1 =$ the relational structure of L_2. Let X be a set. Then

(i) if sup X exists in L_1, then sup X exists in L_2, and

(ii) if inf X exists in L_1, then inf X exists in L_2.

(15) Let L be an antisymmetric relational structure and X be a set. Then sup X exists in L if and only if there exists an element a of L such that $X \leq a$ and for every element b of L such that $X \leq b$ holds $a \leq b$.

(16) Let L be an antisymmetric relational structure and X be a set. Then inf X exists in L if and only if there exists an element a of L such that $X \geq a$ and for every element b of L such that $X \geq b$ holds $a \geq b$.

(17) Let L be a complete non empty antisymmetric relational structure and X be a set. Then sup X exists in L and inf X exists in L.

(18) Let L be an antisymmetric relational structure and a, b, c be elements of L. Then $c = a \sqcup b$ and sup $\{a, b\}$ exists in L if and only if $c \geq a$ and $c \geq b$ and for every element d of L such that $d \geq a$ and $d \geq b$ holds $c \leq d$.

(19) Let L be an antisymmetric relational structure and a, b, c be elements of L. Then $c = a \sqcap b$ and $\inf \{a,b\}$ exists in L if and only if $c \leq a$ and $c \leq b$ and for every element d of L such that $d \leq a$ and $d \leq b$ holds $c \geq d$.

(20) Let L be an antisymmetric relational structure. Then L has l.u.b.'s if and only if for all elements a, b of L holds $\sup \{a,b\}$ exists in L.

(21) Let L be an antisymmetric relational structure. Then L has g.l.b.'s if and only if for all elements a, b of L holds $\inf \{a,b\}$ exists in L.

(22) Let L be an antisymmetric relational structure with l.u.b.'s and a, b, c be elements of L. Then $c = a \sqcup b$ if and only if the following conditions are satisfied:

(i) $c \geq a$,

(ii) $c \geq b$, and

(iii) for every element d of L such that $d \geq a$ and $d \geq b$ holds $c \leq d$.

(23) Let L be an antisymmetric relational structure with g.l.b.'s and a, b, c be elements of L. Then $c = a \sqcap b$ if and only if the following conditions are satisfied:

(i) $c \leq a$,

(ii) $c \leq b$, and

(iii) for every element d of L such that $d \leq a$ and $d \leq b$ holds $c \geq d$.

(24) Let L be an antisymmetric reflexive relational structure with l.u.b.'s and a, b be elements of L. Then $a = a \sqcup b$ if and only if $a \geq b$.

(25) Let L be an antisymmetric reflexive relational structure with g.l.b.'s and a, b be elements of L. Then $a = a \sqcap b$ if and only if $a \leq b$.

Let L be a relational structure and let X be a set. The functor $\bigsqcup_L X$ yielding an element of L is defined as follows:

(Def. 9) $X \leq \bigsqcup_L X$ and for every element a of L such that $X \leq a$ holds $\bigsqcup_L X \leq a$ if $\sup X$ exists in L.

The functor $\bigsqcap_L X$ yields an element of L and is defined by:

(Def. 10) $X \geq \bigsqcap_L X$ and for every element a of L such that $X \geq a$ holds $a \leq \bigsqcap_L X$ if $\inf X$ exists in L.

Next we state a number of propositions:

(26) Let L_1, L_2 be relational structures. Suppose the relational structure of $L_1 =$ the relational structure of L_2. Let X be a set. If $\sup X$ exists in L_1, then $\bigsqcup_{L_1} X = \bigsqcup_{L_2} X$.

(27) Let L_1, L_2 be relational structures. Suppose the relational structure of $L_1 =$ the relational structure of L_2. Let X be a set. If $\inf X$ exists in L_1, then $\bigsqcap_{L_1} X = \bigsqcap_{L_2} X$.

(28) For every complete non empty poset L and for every set X holds $\bigsqcup_L X = \bigsqcup_{(\mathbb{L}_L)} X$ and $\bigsqcap_L X = \bigsqcap_{(\mathbb{L}_L)} X$.

(29) For every complete lattice L and for every set X holds $\bigsqcup_L X = \bigsqcup_{\mathrm{Poset}(L)} X$ and $\bigsqcap_L X = \bigsqcap_{\mathrm{Poset}(L)} X$.

(30) Let L be an antisymmetric relational structure, a be an element of L, and X be a set. Then $a = \bigsqcup_L X$ and $\sup X$ exists in L if and only if $a \geq X$ and for every element b of L such that $b \geq X$ holds $a \leq b$.

(31) Let L be an antisymmetric relational structure, a be an element of L, and X be a set. Then $a = \bigsqcap_L X$ and $\inf X$ exists in L if and only if $a \leq X$ and for every element b of L such that $b \leq X$ holds $a \geq b$.

(32) Let L be a complete antisymmetric non empty relational structure, a be an element of L, and X be a set. Then $a = \bigsqcup_L X$ if and only if the following conditions are satisfied:

(i) $a \geq X$, and

(ii) for every element b of L such that $b \geq X$ holds $a \leq b$.

(33) Let L be a complete antisymmetric non empty relational structure, a be an element of L, and X be a set. Then $a = \bigsqcap_L X$ if and only if the following conditions are satisfied:

(i) $a \leq X$, and

(ii) for every element b of L such that $b \leq X$ holds $a \geq b$.

(34) Let L be a relational structure and X, Y be sets. Suppose $X \subseteq Y$ and sup X exists in L and sup Y exists in L. Then $\bigsqcup_L X \leq \bigsqcup_L Y$.

(35) Let L be a relational structure and X, Y be sets. Suppose $X \subseteq Y$ and inf X exists in L and inf Y exists in L. Then $\bigsqcap_L X \geq \bigsqcap_L Y$.

(36) Let L be an antisymmetric transitive relational structure and X, Y be sets. Suppose sup X exists in L and sup Y exists in L and sup $X \cup Y$ exists in L. Then $\bigsqcup_L (X \cup Y) = \bigsqcup_L X \sqcup \bigsqcup_L Y$.

(37) Let L be an antisymmetric transitive relational structure and X, Y be sets. Suppose inf X exists in L and inf Y exists in L and inf $X \cup Y$ exists in L. Then $\bigsqcap_L (X \cup Y) = \bigsqcap_L X \sqcap \bigsqcap_L Y$.

Let L be a relational structure and let X be a subset of L. We introduce $\sup X$ as a synonym of $\bigsqcup_L X$. We introduce $\inf X$ as a synonym of $\bigsqcap_L X$.

The following propositions are true:

(38) Let L be a non empty reflexive antisymmetric relational structure and a be an element of L. Then sup $\{a\}$ exists in L and inf $\{a\}$ exists in L.

(39) Let L be a non empty reflexive antisymmetric relational structure and a be an element of L. Then $\sup\{a\} = a$ and $\inf\{a\} = a$.

(40) For every poset L with g.l.b.'s and for all elements a, b of L holds $\inf\{a, b\} = a \sqcap b$.

(41) For every poset L with l.u.b.'s and for all elements a, b of L holds $\sup\{a, b\} = a \sqcup b$.

(42) Let L be a lower-bounded antisymmetric non empty relational structure. Then sup $\emptyset$ exists in L and inf the carrier of L exists in L.

(43) Let L be an upper-bounded antisymmetric non empty relational structure. Then inf $\emptyset$ exists in L and sup the carrier of L exists in L.

Let L be a relational structure. The functor $\perp_L$ yielding an element of L is defined as follows:

(Def. 11) $\perp_L = \bigsqcup_L \emptyset$.

The functor $\top_L$ yielding an element of L is defined as follows:

(Def. 12) $\top_L = \bigsqcap_L \emptyset$.

Next we state a number of propositions:

(44) For every lower-bounded antisymmetric non empty relational structure L and for every element x of L holds $\perp_L \leq x$.

(45) For every upper-bounded antisymmetric non empty relational structure L and for every element x of L holds $x \leq \top_L$.

(46) Let L be a non empty relational structure and X, Y be sets. Suppose that for every element x of L holds $x \geq X$ iff $x \geq Y$. If sup X exists in L, then sup Y exists in L.

(47) Let L be a non empty relational structure and X, Y be sets. Suppose sup X exists in L and for every element x of L holds $x \geq X$ iff $x \geq Y$. Then $\bigsqcup_L X = \bigsqcup_L Y$.

(48) Let L be a non empty relational structure and X, Y be sets. Suppose that for every element x of L holds $x \leq X$ iff $x \leq Y$. If inf X exists in L, then inf Y exists in L.

(49) Let L be a non empty relational structure and X, Y be sets. Suppose inf X exists in L and for every element x of L holds $x \leq X$ iff $x \leq Y$. Then $\bigsqcap_L X = \bigsqcap_L Y$.

(50) Let L be a non empty relational structure and X be a set. Then

(i) sup X exists in L iff sup $X \cap$ the carrier of L exists in L, and

(ii) inf X exists in L iff inf $X \cap$ the carrier of L exists in L.

(51) Let L be a non empty relational structure and X be a set. Suppose sup X exists in L or sup $X \cap$ the carrier of L exists in L. Then $\bigsqcup_L X = \bigsqcup_L (X \cap \text{the carrier of } L)$.

(52) Let L be a non empty relational structure and X be a set. Suppose inf X exists in L or inf $X \cap$ the carrier of L exists in L. Then $\bigsqcap_L X = \bigsqcap_L (X \cap \text{the carrier of } L)$.

(53) Let L be a non empty relational structure. If for every subset X of L holds sup X exists in L, then L is complete.

(54) Let L be a non empty poset. Then L has l.u.b.'s if and only if for every finite non empty subset X of L holds sup X exists in L.

(55) Let L be a non empty poset. Then L has g.l.b.'s if and only if for every finite non empty subset X of L holds inf X exists in L.

3. Relational substructures

We now state the proposition

(56) For every set X and for every binary relation R on X holds $R = R \mid^2 X$.

Let L be a relational structure. A relational structure is said to be a relational substructure of L if:

(Def. 13) The carrier of it $\subseteq$ the carrier of L and the internal relation of it $\subseteq$ the internal relation of L.

Let L be a relational structure and let S be a relational substructure of L. We say that S is full if and only if:

(Def. 14) The internal relation of $S =$ (the internal relation of L) $\mid^2$ (the carrier of S).

Let L be a relational structure. Observe that there exists a relational substructure of L which is strict and full.

Let L be a non empty relational structure. One can verify that there exists a relational substructure of L which is non empty, full, and strict.

Next we state two propositions:

(57) Let L be a relational structure and X be a subset of L. Then $\langle X, (\text{the internal relation of } L) \mid^2 X \rangle$ is a full relational substructure of L.

(58) Let L be a relational structure and S_1, S_2 be full relational substructures of L. Suppose the carrier of $S_1 =$ the carrier of S_2. Then the relational structure of $S_1 =$ the relational structure of S_2.

Let L be a relational structure and let X be a subset of L. The functor sub(X) yields a full strict relational substructure of L and is defined by:

(Def. 15) The carrier of sub$(X) = X$.

The following propositions are true:

(59) Let L be a non empty relational structure and S be a non empty relational substructure of L. Then every element of S is an element of L.

(60) Let L be a relational structure, S be a relational substructure of L, a, b be elements of L, and x, y be elements of S. If $x = a$ and $y = b$ and $x \leq y$, then $a \leq b$.

(61) Let L be a relational structure, S be a full relational substructure of L, a, b be elements of L, and x, y be elements of S. Suppose $x = a$ and $y = b$ and $a \leq b$ and $x \in$ the carrier of S and $y \in$ the carrier of S. Then $x \leq y$.

(62) Let L be a non empty relational structure, S be a non empty full relational substructure of L, X be a set, a be an element of L, and x be an element of S such that $x = a$. Then

(i) if $a \leq X$, then $x \leq X$, and

(ii) if $a \geq X$, then $x \geq X$.

(63) Let L be a non empty relational structure, S be a non empty relational substructure of L, X be a subset of S, a be an element of L, and x be an element of S such that $x = a$. Then

(i) if $x \leq X$, then $a \leq X$, and

(ii) if $x \geq X$, then $a \geq X$.

Let L be a reflexive relational structure. Note that every full relational substructure of L is reflexive.

Let L be a transitive relational structure. One can check that every full relational substructure of L is transitive.

Let L be an antisymmetric relational structure. Observe that every full relational substructure of L is antisymmetric.

Let L be a non empty relational structure and let S be a relational substructure of L. We say that S is meet-inheriting if and only if the condition (Def. 16) is satisfied.

(Def. 16) Let x, y be elements of L. Suppose $x \in$ the carrier of S and $y \in$ the carrier of S and inf $\{x,y\}$ exists in L. Then $\inf\{x,y\} \in$ the carrier of S.

We say that S is join-inheriting if and only if the condition (Def. 17) is satisfied.

(Def. 17) Let x, y be elements of L. Suppose $x \in$ the carrier of S and $y \in$ the carrier of S and sup $\{x,y\}$ exists in L. Then $\sup\{x,y\} \in$ the carrier of S.

Let L be a non empty relational structure and let S be a relational substructure of L. We say that S is infs-inheriting if and only if:

(Def. 18) For every subset X of S such that inf X exists in L holds $\bigsqcap_L X \in$ the carrier of S.

We say that S is sups-inheriting if and only if:

(Def. 19) For every subset X of S such that sup X exists in L holds $\bigsqcup_L X \in$ the carrier of S.

Let L be a non empty relational structure. Observe that every relational substructure of L which is infs-inheriting is also meet-inheriting and every relational substructure of L which is sups-inheriting is also join-inheriting.

Let L be a non empty relational structure. One can verify that there exists a relational substructure of L which is infs-inheriting, sups-inheriting, non empty, full, and strict.

One can prove the following four propositions:

(64) Let L be a non empty transitive relational structure, S be a non empty full relational substructure of L, and X be a subset of S. Suppose inf X exists in L and $\bigsqcap_L X \in$ the carrier of S. Then inf X exists in S and $\bigsqcap_S X = \bigsqcap_L X$.

(65) Let L be a non empty transitive relational structure, S be a non empty full relational substructure of L, and X be a subset of S. Suppose sup X exists in L and $\bigsqcup_L X \in$ the carrier of S. Then sup X exists in S and $\bigsqcup_S X = \bigsqcup_L X$.

(66) Let L be a non empty transitive relational structure, S be a non empty full relational substructure of L, and x, y be elements of S. Suppose inf $\{x,y\}$ exists in L and $\bigsqcap_L\{x,y\} \in$ the carrier of S. Then inf $\{x,y\}$ exists in S and $\bigsqcap_S\{x,y\} = \bigsqcap_L\{x,y\}$.

(67) Let L be a non empty transitive relational structure, S be a non empty full relational substructure of L, and x, y be elements of S. Suppose sup $\{x,y\}$ exists in L and $\bigsqcup_L\{x,y\} \in$ the carrier of S. Then sup $\{x,y\}$ exists in S and $\bigsqcup_S\{x,y\} = \bigsqcup_L\{x,y\}$.

Let L be an antisymmetric transitive relational structure with g.l.b.'s. Observe that every non empty meet-inheriting full relational substructure of L has g.l.b.'s.

Let L be an antisymmetric transitive relational structure with l.u.b.'s. Note that every non empty join-inheriting full relational substructure of L has l.u.b.'s.

The following propositions are true:

(68) Let L be a complete non empty poset, S be a non empty full relational substructure of L, and X be a subset of S. If $\bigsqcap_L X \in$ the carrier of S, then $\bigsqcap_S X = \bigsqcap_L X$.

(69) Let L be a complete non empty poset, S be a non empty full relational substructure of L, and X be a subset of S. If $\bigsqcup_L X \in$ the carrier of S, then $\bigsqcup_S X = \bigsqcup_L X$.

(70) Let L be a poset with g.l.b.'s, S be a meet-inheriting non empty full relational substructure of L, x, y be elements of S, and a, b be elements of L. If $a = x$ and $b = y$, then $x \sqcap y = a \sqcap b$.

(71) Let L be a poset with l.u.b.'s, S be a join-inheriting non empty full relational substructure of L, x, y be elements of S, and a, b be elements of L. If $a = x$ and $b = y$, then $x \sqcup y = a \sqcup b$.

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