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On the Characterizations of Compactness

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Summary. In the paper we show equivalence of the convergence of filters on a topological space and the convergence of nets in the space. We also give, five characterizations of compactness. Namely, for any topological space T we proved that following condition are equivalent:

- T is compact,
- every ultrafilter on T is convergent,
- every proper filter on T has cluster point,
- every net in T has cluster point,
- every net in T has convergent subnet,
- every Cauchy net in T is convergent.

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The articles [18], [7], [22], [23], [19], [14], [10], [5], [25], [24], [6], [16], [9], [12], [8], [15], [17], [21], [1], [2], [3], [11], [4], [20], and [13] provide the notation and terminology for this paper.

One can prove the following proposition

(2)[1] For every non empty set X and for every proper filter F of $2^{X}_{\subseteq}$ and for every set A such that $A \in F$ holds A is not empty.

Let T be a non empty topological space and let x be a point of T. The neighborhood system of x is a subset of $2^{\Omega_T}_{\subseteq}$ and is defined by:

(Def. 1) The neighborhood system of $x = \{A : A$ ranges over neighbourhoods of $x\}$.

The following proposition is true

(3) Let T be a non empty topological space, x be a point of T, and A be a set. Then $A \in$ the neighborhood system of x if and only if A is a neighbourhood of x.

Let T be a non empty topological space and let x be a point of T. Note that the neighborhood system of x is non empty, proper, upper, and filtered.

The following propositions are true:

(4) Let T be a non empty topological space, x be a point of T, and F be an upper subset of $2^{\Omega_T}_{\subseteq}$. Then x is a convergence point of F, T if and only if the neighborhood system of $x \subseteq F$.

[1] The proposition (1) has been removed.

(5) For every non empty topological space T holds every point x of T is a convergence point of the neighborhood system of x, T.

(6) Let T be a non empty topological space and A be a subset of T. Then A is open if and only if for every point x of T such that $x \in A$ and for every filter F of $2^{\Omega_T}_{\subseteq}$ such that x is a convergence point of F, T holds $A \in F$.

Let S be a non empty 1-sorted structure and let N be a non empty net structure over S. A subset of S is called a subset of S reachable by N if:

(Def. 2) There exists an element i of N such that it $=$ rng(the mapping of $N{\restriction}i$).

Next we state the proposition

(7) Let S be a non empty 1-sorted structure, N be a non empty net structure over S, and i be an element of N. Then rng(the mapping of $N{\restriction}i$) is a subset of S reachable by N.

Let S be a non empty 1-sorted structure and let N be a reflexive non empty net structure over S. Observe that every subset of S reachable by N is non empty.

The following three propositions are true:

(8) Let S be a non empty 1-sorted structure, N be a net in S, i be an element of N, and x be a set. Then $x \in$ rng(the mapping of $N{\restriction}i$) if and only if there exists an element j of N such that $i \leq j$ and $x = N(j)$.

(9) Let S be a non empty 1-sorted structure, N be a net in S, and A be a subset of S reachable by N. Then N is eventually in A.

(10) Let S be a non empty 1-sorted structure, N be a net in S, and F be a finite non empty set. Suppose every element of F is a subset of S reachable by N. Then there exists a subset B of S reachable by N such that $B \subseteq \bigcap F$.

Let T be a non empty 1-sorted structure and let N be a non empty net structure over T. The filter of N is a subset of $2^{\Omega_T}_{\subseteq}$ and is defined as follows:

(Def. 3) The filter of $N = \{A; A$ ranges over subsets of T: N is eventually in $A\}$.

We now state the proposition

(11) Let T be a non empty 1-sorted structure, N be a non empty net structure over T, and A be a set. Then $A \in$ the filter of N if and only if N is eventually in A and A is a subset of T.

Let T be a non empty 1-sorted structure and let N be a non empty net structure over T. One can check that the filter of N is non empty and upper.

Let T be a non empty 1-sorted structure and let N be a net in T. One can verify that the filter of N is proper and filtered.

One can prove the following propositions:

(12) Let T be a non empty topological space, N be a net in T, and x be a point of T. Then x is a cluster point of N if and only if x is a cluster point of the filter of N, T.

(13) Let T be a non empty topological space, N be a net in T, and x be a point of T. Then $x \in \mathrm{Lim} N$ if and only if x is a convergence point of the filter of N, T.

Let L be a non empty 1-sorted structure, let O be a non empty subset of L, and let F be a filter of $2^{O}_{\subseteq}$. The net of F is a strict non empty net structure over L and is defined by the conditions (Def. 4).

(Def. 4)(i) The carrier of the net of $F = \{\langle a, f\rangle; a$ ranges over elements of L, f ranges over elements of F: $a \in f\}$,

(ii) for all elements i, j of the net of F holds $i \leq j$ iff $j_{\mathbf{2}} \subseteq i_{\mathbf{2}}$, and

(iii) for every element i of the net of F holds (the net of $F)(i) = i_{\mathbf{1}}$.

Let L be a non empty 1-sorted structure, let O be a non empty subset of L, and let F be a filter of $2^{O}_{\subseteq}$. Observe that the net of F is reflexive and transitive.

Let L be a non empty 1-sorted structure, let O be a non empty subset of L, and let F be a proper filter of $2^{O}_{\subseteq}$. Note that the net of F is directed.

One can prove the following propositions:

(14) For every non empty 1-sorted structure T and for every filter F of $2^{\Omega_T}_{\subseteq}$ holds $F \setminus \{\emptyset\} =$ the filter of the net of F.

(15) Let T be a non empty 1-sorted structure and F be a proper filter of $2^{\Omega_T}_{\subseteq}$. Then $F =$ the filter of the net of F.

(16) Let T be a non empty 1-sorted structure, F be a filter of $2^{\Omega_T}_{\subseteq}$, and A be a non empty subset of T. Then $A \in F$ if and only if the net of F is eventually in A.

(17) Let T be a non empty topological space, F be a proper filter of $2^{\Omega_T}_{\subseteq}$, and x be a point of T. Then x is a cluster point of the net of F if and only if x is a cluster point of F, T.

(18) Let T be a non empty topological space, F be a proper filter of $2^{\Omega_T}_{\subseteq}$, and x be a point of T. Then $x \in \mathrm{Lim}(\text{the net of } F)$ if and only if x is a convergence point of F, T.

(20)[2] Let T be a non empty topological space, x be a point of T, and A be a subset of T. Suppose $x \in \overline{A}$. Let F be a proper filter of $2^{\Omega_T}_{\subseteq}$. If $F =$ the neighborhood system of x, then the net of F is often in A.

(21) Let T be a non empty 1-sorted structure, A be a set, and N be a net in T. If N is eventually in A, then every subnet of N is eventually in A.

(22) Let T be a non empty topological space and F, G, x be sets. Suppose $F \subseteq G$ and x is a convergence point of F, T. Then x is a convergence point of G, T.

(23) Let T be a non empty topological space, A be a subset of T, and x be a point of T. Then $x \in \overline{A}$ if and only if there exists a net N in T such that N is eventually in A and x is a cluster point of N.

(24) Let T be a non empty topological space, A be a subset of T, and x be a point of T. Then $x \in \overline{A}$ if and only if there exists a convergent net N in T such that N is eventually in A and $x \in \mathrm{Lim}N$.

(25) Let T be a non empty topological space and A be a subset of T. Then A is closed if and only if for every net N in T such that N is eventually in A and for every point x of T such that x is a cluster point of N holds $x \in A$.

(26) Let T be a non empty topological space and A be a subset of T. Then A is closed if and only if for every convergent net N in T such that N is eventually in A and for every point x of T such that $x \in \mathrm{Lim}N$ holds $x \in A$.

(27) Let T be a non empty topological space, A be a subset of T, and x be a point of T. Then $x \in \overline{A}$ if and only if there exists a proper filter F of $2^{\Omega_T}_{\subseteq}$ such that $A \in F$ and x is a cluster point of F, T.

(28) Let T be a non empty topological space, A be a subset of T, and x be a point of T. Then $x \in \overline{A}$ if and only if there exists an ultra filter F of $2^{\Omega_T}_{\subseteq}$ such that $A \in F$ and x is a convergence point of F, T.

(29) Let T be a non empty topological space and A be a subset of T. Then A is closed if and only if for every proper filter F of $2^{\Omega_T}_{\subseteq}$ such that $A \in F$ and for every point x of T such that x is a cluster point of F, T holds $x \in A$.

[2] The proposition (19) has been removed.

(30) Let T be a non empty topological space and A be a subset of T. Then A is closed if and only if for every ultra filter F of $2^{\Omega_T}_{\subseteq}$ such that $A \in F$ and for every point x of T such that x is a convergence point of F, T holds $x \in A$.

(31) Let T be a non empty topological space, N be a net in T, and s be a point of T. Then s is a cluster point of N if and only if for every subset A of T reachable by N holds $s \in \overline{A}$.

(32) For every non empty topological space T and for every family F of subsets of T such that F is closed holds $\mathrm{FinMeetCl}(F)$ is closed.

(33) Let T be a non empty topological space. Then T is compact if and only if for every ultra filter F of $2^{\Omega_T}_{\subseteq}$ holds there exists a point of T which is a convergence point of F, T.

(34) Let T be a non empty topological space. Then T is compact if and only if for every proper filter F of $2^{\Omega_T}_{\subseteq}$ holds there exists a point of T which is a cluster point of F, T.

(35) Let T be a non empty topological space. Then T is compact if and only if for every net N in T holds there exists a point of T which is a cluster point of N.

(36) Let T be a non empty topological space. Then T is compact if and only if for every net N in T such that $N \in \mathrm{NetUniv}(T)$ holds there exists a point of T which is a cluster point of N.

Let L be a non empty 1-sorted structure and let N be a transitive net structure over L. Note that every full structure of a subnet of N is transitive.

Let L be a non empty 1-sorted structure and let N be a non empty directed net structure over L. Note that there exists a structure of a subnet of N which is strict, non empty, directed, and full.

Next we state the proposition

(37) For every non empty topological space T holds T is compact iff for every net N in T holds there exists a subnet of N which is convergent.

Let S be a non empty 1-sorted structure and let N be a non empty net structure over S. We say that N is Cauchy if and only if:

(Def. 5) For every subset A of S holds N is eventually in A and eventually in A^{c}.

Let S be a non empty 1-sorted structure and let F be an ultra filter of $2^{\Omega_S}_{\subseteq}$. One can check that the net of F is Cauchy.

One can prove the following proposition

(38) Let T be a non empty topological space. Then T is compact if and only if for every net N in T such that N is Cauchy holds N is convergent.

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