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Concrete Categories

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Summary. In the paper, we develop the notation of duality and equivalence of categories and concrete categories based on [22]. The development was motivated by the duality theory for continuous lattices (see [10, p. 189]), where we need to cope with concrete categories of lattices and maps preserving their properties. For example, the category *UPS* of complete lattices and directed suprema preserving maps; or the category *INF* of complete lattices and infima preserving maps. As the main result of this paper it is shown that every category is isomorphic to its concretization (the concrete category with the same objects). Some useful schemes to construct categories and functors are also presented.

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The articles [18], [9], [26], [25], [27], [28], [6], [8], [7], [16], [19], [5], [15], [2], [1], [20], [13], [21], [12], [4], [3], [22], [23], [24], [14], [17], and [11] provide the notation and terminology for this paper.

1. Definability of Categories and Functors

In this article we present several logical schemes. The scheme *AltCatStrLambda* deals with a non empty set $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, and a 5-ary functor $\mathcal{G}$ yielding a set, and states that:

There exists a strict non empty transitive category structure C such that

(i) the carrier of $C = \mathcal{A}$,

(ii) for all objects a, b of C holds $\langle a,b\rangle = \mathcal{F}(a,b)$, and

(iii) for all objects a, b, c of C such that $\langle a,b\rangle \neq \emptyset$ and $\langle b,c\rangle \neq \emptyset$ and for every morphism f from a to b and for every morphism g from b to c holds $g \cdot f = \mathcal{G}(a,b,c,f,g)$

provided the following requirement is met:

- For all elements a, b, c of $\mathcal{A}$ and for all sets f, g such that $f \in \mathcal{F}(a,b)$ and $g \in \mathcal{F}(b,c)$ holds $\mathcal{G}(a,b,c,f,g) \in \mathcal{F}(a,c)$.

The scheme *CatAssocSch* deals with a non empty transitive category structure $\mathcal{A}$ and a 5-ary functor $\mathcal{F}$ yielding a set, and states that:

$\mathcal{A}$ is associative

provided the parameters satisfy the following conditions:

- Let a, b, c be objects of $\mathcal{A}$. Suppose $\langle a,b\rangle \neq \emptyset$ and $\langle b,c\rangle \neq \emptyset$. Let f be a morphism from a to b and g be a morphism from b to c. Then $g \cdot f = \mathcal{F}(a,b,c,f,g)$, and
- Let a, b, c, d be objects of $\mathcal{A}$ and f, g, h be sets. If $f \in \langle a,b\rangle$ and $g \in \langle b,c\rangle$ and $h \in \langle c,d\rangle$, then $\mathcal{F}(a,c,d,\mathcal{F}(a,b,c,f,g),h) = \mathcal{F}(a,b,d,f,\mathcal{F}(b,c,d,g,h))$.

The scheme *CatUnitsSch* deals with a non empty transitive category structure $\mathcal{A}$ and a 5-ary functor $\mathcal{F}$ yielding a set, and states that:

$\mathcal{A}$ has units

provided the parameters satisfy the following conditions:

- Let a, b, c be objects of $\mathcal{A}$. Suppose $\langle a,b\rangle \neq \emptyset$ and $\langle b,c\rangle \neq \emptyset$. Let f be a morphism from a to b and g be a morphism from b to c. Then $g \cdot f = \mathcal{F}(a,b,c,f,g)$,
- Let a be an object of $\mathcal{A}$. Then there exists a set f such that $f \in \langle a,a\rangle$ and for every object b of $\mathcal{A}$ and for every set g such that $g \in \langle a,b\rangle$ holds $\mathcal{F}(a,a,b,f,g) = g$, and
- Let a be an object of $\mathcal{A}$. Then there exists a set f such that $f \in \langle a,a\rangle$ and for every object b of $\mathcal{A}$ and for every set g such that $g \in \langle b,a\rangle$ holds $\mathcal{F}(b,a,a,g,f) = g$.

The scheme *CategoryLambda* deals with a non empty set $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, and a 5-ary functor $\mathcal{G}$ yielding a set, and states that:

There exists a strict category C such that
(i) the carrier of $C = \mathcal{A}$,
(ii) for all objects a, b of C holds $\langle a,b\rangle = \mathcal{F}(a,b)$, and
(iii) for all objects a, b, c of C such that $\langle a,b\rangle \neq \emptyset$ and $\langle b,c\rangle \neq \emptyset$ and for every morphism f from a to b and for every morphism g from b to c holds $g \cdot f = \mathcal{G}(a,b,c,f,g)$

provided the following conditions are met:

- For all elements a, b, c of $\mathcal{A}$ and for all sets f, g such that $f \in \mathcal{F}(a,b)$ and $g \in \mathcal{F}(b,c)$ holds $\mathcal{G}(a,b,c,f,g) \in \mathcal{F}(a,c)$,
- Let a, b, c, d be elements of $\mathcal{A}$ and f, g, h be sets. If $f \in \mathcal{F}(a,b)$ and $g \in \mathcal{F}(b,c)$ and $h \in \mathcal{F}(c,d)$, then $\mathcal{G}(a,c,d,\mathcal{G}(a,b,c,f,g),h) = \mathcal{G}(a,b,d,f,\mathcal{G}(b,c,d,g,h))$,
- Let a be an element of $\mathcal{A}$. Then there exists a set f such that $f \in \mathcal{F}(a,a)$ and for every element b of $\mathcal{A}$ and for every set g such that $g \in \mathcal{F}(a,b)$ holds $\mathcal{G}(a,a,b,f,g) = g$, and
- Let a be an element of $\mathcal{A}$. Then there exists a set f such that $f \in \mathcal{F}(a,a)$ and for every element b of $\mathcal{A}$ and for every set g such that $g \in \mathcal{F}(b,a)$ holds $\mathcal{G}(b,a,a,g,f) = g$.

The scheme *CategoryLambdaUniq* deals with a non empty set $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, and a 5-ary functor $\mathcal{G}$ yielding a set, and states that:

Let C_1, C_2 be non empty transitive category structures. Suppose that
(i) the carrier of $C_1 = \mathcal{A}$,
(ii) for all objects a, b of C_1 holds $\langle a,b\rangle = \mathcal{F}(a,b)$,
(iii) for all objects a, b, c of C_1 such that $\langle a,b\rangle \neq \emptyset$ and $\langle b,c\rangle \neq \emptyset$ and for every morphism f from a to b and for every morphism g from b to c holds $g \cdot f = \mathcal{G}(a,b,c,f,g)$,
(iv) the carrier of $C_2 = \mathcal{A}$,
(v) for all objects a, b of C_2 holds $\langle a,b\rangle = \mathcal{F}(a,b)$, and
(vi) for all objects a, b, c of C_2 such that $\langle a,b\rangle \neq \emptyset$ and $\langle b,c\rangle \neq \emptyset$ and for every morphism f from a to b and for every morphism g from b to c holds $g \cdot f = \mathcal{G}(a,b,c,f,g)$.

Then the category structure of $C_1 =$ the category structure of C_2

for all values of the parameters.

The scheme *CategoryQuasiLambda* deals with a non empty set $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, a 5-ary functor $\mathcal{G}$ yielding a set, and a ternary predicate $\mathcal{P}$, and states that:

There exists a strict category C such that
(i) the carrier of $C = \mathcal{A}$,
(ii) for all objects a, b of C and for every set f holds $f \in \langle a,b\rangle$ iff $f \in \mathcal{F}(a,b)$ and $\mathcal{P}[a,b,f]$, and
(iii) for all objects a, b, c of C such that $\langle a,b\rangle \neq \emptyset$ and $\langle b,c\rangle \neq \emptyset$ and for every morphism f from a to b and for every morphism g from b to c holds $g \cdot f = \mathcal{G}(a,b,c,f,g)$

provided the parameters meet the following requirements:

- Let a, b, c be elements of $\mathcal{A}$ and f, g be sets. Suppose $f \in \mathcal{F}(a,b)$ and $\mathcal{P}[a,b,f]$ and $g \in \mathcal{F}(b,c)$ and $\mathcal{P}[b,c,g]$. Then $\mathcal{G}(a,b,c,f,g) \in \mathcal{F}(a,c)$ and $\mathcal{P}[a,c,\mathcal{G}(a,b,c,f,g)]$,
- Let a, b, c, d be elements of $\mathcal{A}$ and f, g, h be sets. Suppose $f \in \mathcal{F}(a,b)$ and $\mathcal{P}[a,b,f]$ and $g \in \mathcal{F}(b,c)$ and $\mathcal{P}[b,c,g]$ and $h \in \mathcal{F}(c,d)$ and $\mathcal{P}[c,d,h]$. Then $\mathcal{G}(a,c,d,\mathcal{G}(a,b,c,f,g),h) = \mathcal{G}(a,b,d,f,\mathcal{G}(b,c,d,g,h))$,
- Let a be an element of $\mathcal{A}$. Then there exists a set f such that $f \in \mathcal{F}(a,a)$ and $\mathcal{P}[a,a,f]$ and for every element b of $\mathcal{A}$ and for every set g such that $g \in \mathcal{F}(a,b)$ and $\mathcal{P}[a,b,g]$ holds $\mathcal{G}(a,a,b,f,g) = g$, and

- Let a be an element of $\mathcal{A}$. Then there exists a set f such that $f \in \mathcal{F}(a,a)$ and $\mathcal{P}[a,a,f]$ and for every element b of $\mathcal{A}$ and for every set g such that $g \in \mathcal{F}(b,a)$ and $\mathcal{P}[b,a,g]$ holds $\mathcal{G}(b,a,a,g,f) = g$.

Let f be a function yielding function and let a, b, c be sets. Note that $f(a, b, c)$ is relation-like and function-like.

Now we present two schemes. The scheme *SubcategoryEx* deals with a category $\mathcal{A}$, a unary predicate $\mathcal{P}$, and a ternary predicate $\mathcal{Q}$, and states that:

There exists a strict non empty subcategory B of $\mathcal{A}$ such that
(i) for every object a of $\mathcal{A}$ holds a is an object of B iff $\mathcal{P}[a]$, and
(ii) for all objects a, b of $\mathcal{A}$ and for all objects a', b' of B such that $a' = a$ and $b' = b$ and $\langle a,b \rangle \neq \emptyset$ and for every morphism f from a to b holds $f \in \langle a',b' \rangle$ iff $\mathcal{Q}[a,b,f]$

provided the following conditions are met:

- There exists an object a of $\mathcal{A}$ such that $\mathcal{P}[a]$,
- Let a, b, c be objects of $\mathcal{A}$. Suppose $\mathcal{P}[a]$ and $\mathcal{P}[b]$ and $\mathcal{P}[c]$ and $\langle a,b \rangle \neq \emptyset$ and $\langle b,c \rangle \neq \emptyset$. Let f be a morphism from a to b and g be a morphism from b to c. If $\mathcal{Q}[a,b,f]$ and $\mathcal{Q}[b,c,g]$, then $\mathcal{Q}[a,c,g \cdot f]$, and
- For every object a of $\mathcal{A}$ such that $\mathcal{P}[a]$ holds $\mathcal{Q}[a,a,\mathrm{id}_a]$.

The scheme *CovariantFunctorLambda* deals with categories $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a ternary functor $\mathcal{G}$ yielding a set, and states that:

There exists a covariant strict functor F from $\mathcal{A}$ to $\mathcal{B}$ such that
(i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(a)$, and
(ii) for all objects a, b of $\mathcal{A}$ such that $\langle a,b \rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{G}(a,b,f)$

provided the parameters meet the following conditions:

- For every object a of $\mathcal{A}$ holds $\mathcal{F}(a)$ is an object of $\mathcal{B}$,
- Let a, b be objects of $\mathcal{A}$. Suppose $\langle a,b \rangle \neq \emptyset$. Let f be a morphism from a to b. Then $\mathcal{G}(a,b,f) \in$ (the arrows of $\mathcal{B}$)($\mathcal{F}(a)$, $\mathcal{F}(b)$),
- Let a, b, c be objects of $\mathcal{A}$. Suppose $\langle a,b \rangle \neq \emptyset$ and $\langle b,c \rangle \neq \emptyset$. Let f be a morphism from a to b, g be a morphism from b to c, and a', b', c' be objects of $\mathcal{B}$. Suppose $a' = \mathcal{F}(a)$ and $b' = \mathcal{F}(b)$ and $c' = \mathcal{F}(c)$. Let f' be a morphism from a' to b' and g' be a morphism from b' to c'. If $f' = \mathcal{G}(a,b,f)$ and $g' = \mathcal{G}(b,c,g)$, then $\mathcal{G}(a,c,g \cdot f) = g' \cdot f'$, and
- For every object a of $\mathcal{A}$ and for every object a' of $\mathcal{B}$ such that $a' = \mathcal{F}(a)$ holds $\mathcal{G}(a,a,\mathrm{id}_a) = \mathrm{id}_{a'}$.

One can prove the following proposition

(1) Let A, B be categories and F, G be covariant functors from A to B. Suppose that

(i) for every object a of A holds $F(a) = G(a)$, and

(ii) for all objects a, b of A such that $\langle a,b \rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = G(f)$.

Then the functor structure of $F =$ the functor structure of G.

The scheme *ContravariantFunctorLambda* deals with categories $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a ternary functor $\mathcal{G}$ yielding a set, and states that:

There exists a contravariant strict functor F from $\mathcal{A}$ to $\mathcal{B}$ such that
(i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(a)$, and
(ii) for all objects a, b of $\mathcal{A}$ such that $\langle a,b \rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{G}(a,b,f)$

provided the parameters meet the following conditions:

- For every object a of $\mathcal{A}$ holds $\mathcal{F}(a)$ is an object of $\mathcal{B}$,
- Let a, b be objects of $\mathcal{A}$. Suppose $\langle a,b \rangle \neq \emptyset$. Let f be a morphism from a to b. Then $\mathcal{G}(a,b,f) \in$ (the arrows of $\mathcal{B}$)($\mathcal{F}(b)$, $\mathcal{F}(a)$),
- Let a, b, c be objects of $\mathcal{A}$. Suppose $\langle a,b \rangle \neq \emptyset$ and $\langle b,c \rangle \neq \emptyset$. Let f be a morphism from a to b, g be a morphism from b to c, and a', b', c' be objects of $\mathcal{B}$. Suppose

$a' = \mathcal{F}(a)$ and $b' = \mathcal{F}(b)$ and $c' = \mathcal{F}(c)$. Let f' be a morphism from b' to a' and g' be a morphism from c' to b'. If $f' = \mathcal{G}(a,b,f)$ and $g' = \mathcal{G}(b,c,g)$, then $\mathcal{G}(a,c,g\cdot f) = f'\cdot g'$, and
- For every object a of $\mathcal{A}$ and for every object a' of $\mathcal{B}$ such that $a' = \mathcal{F}(a)$ holds $\mathcal{G}(a,a,\mathrm{id}_a) = \mathrm{id}_{a'}$.

Next we state the proposition

(2) Let A, B be categories and F, G be contravariant functors from A to B. Suppose that

(i) for every object a of A holds $F(a) = G(a)$, and

(ii) for all objects a, b of A such that $\langle a,b\rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = G(f)$.

Then the functor structure of $F =$ the functor structure of G.

2. Isomorphism and Equivalence of Categories

Let A, B, C be non empty sets and let f be a function from $[:A, B:]$ into C. Let us observe that f is one-to-one if and only if:

(Def. 1) For all elements a_1, a_2 of A and for all elements b_1, b_2 of B such that $f(a_1, b_1) = f(a_2, b_2)$ holds $a_1 = a_2$ and $b_1 = b_2$.

Now we present four schemes. The scheme *CoBijectiveSch* deals with categories $\mathcal{A}$, $\mathcal{B}$, a covariant functor $\mathcal{C}$ from $\mathcal{A}$ to $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a ternary functor $\mathcal{C}$ yielding a set, and states that:

$\mathcal{C}$ is bijective

provided the following conditions are satisfied:
- For every object a of $\mathcal{A}$ holds $\mathcal{C}(a) = \mathcal{F}(a)$,
- For all objects a, b of $\mathcal{A}$ such that $\langle a,b\rangle \neq \emptyset$ and for every morphism f from a to b holds $\mathcal{C}(f) = \mathcal{C}(a,b,f)$,
- For all objects a, b of $\mathcal{A}$ such that $\mathcal{F}(a) = \mathcal{F}(b)$ holds $a = b$,
- For all objects a, b of $\mathcal{A}$ such that $\langle a,b\rangle \neq \emptyset$ and for all morphisms f, g from a to b such that $\mathcal{C}(a,b,f) = \mathcal{C}(a,b,g)$ holds $f = g$, and
- Let a, b be objects of $\mathcal{B}$. Suppose $\langle a,b\rangle \neq \emptyset$. Let f be a morphism from a to b. Then there exist objects c, d of $\mathcal{A}$ and there exists a morphism g from c to d such that $a = \mathcal{F}(c)$ and $b = \mathcal{F}(d)$ and $\langle c,d\rangle \neq \emptyset$ and $f = \mathcal{C}(c,d,g)$.

The scheme *CatIsomorphism* deals with categories $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a ternary functor $\mathcal{G}$ yielding a set, and states that:

$\mathcal{A}$ and $\mathcal{B}$ are isomorphic

provided the parameters satisfy the following conditions:
- There exists a covariant functor F from $\mathcal{A}$ to $\mathcal{B}$ such that
  (i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(a)$, and
  (ii) for all objects a, b of $\mathcal{A}$ such that $\langle a,b\rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{G}(a,b,f)$,
- For all objects a, b of $\mathcal{A}$ such that $\mathcal{F}(a) = \mathcal{F}(b)$ holds $a = b$,
- For all objects a, b of $\mathcal{A}$ such that $\langle a,b\rangle \neq \emptyset$ and for all morphisms f, g from a to b such that $\mathcal{G}(a,b,f) = \mathcal{G}(a,b,g)$ holds $f = g$, and
- Let a, b be objects of $\mathcal{B}$. Suppose $\langle a,b\rangle \neq \emptyset$. Let f be a morphism from a to b. Then there exist objects c, d of $\mathcal{A}$ and there exists a morphism g from c to d such that $a = \mathcal{F}(c)$ and $b = \mathcal{F}(d)$ and $\langle c,d\rangle \neq \emptyset$ and $f = \mathcal{G}(c,d,g)$.

The scheme *ContraBijectiveSch* deals with categories $\mathcal{A}$, $\mathcal{B}$, a contravariant functor $\mathcal{C}$ from $\mathcal{A}$ to $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a ternary functor $\mathcal{C}$ yielding a set, and states that:

$\mathcal{C}$ is bijective

provided the parameters meet the following requirements:
- For every object a of $\mathcal{A}$ holds $\mathcal{C}(a) = \mathcal{F}(a)$,
- For all objects a, b of $\mathcal{A}$ such that $\langle a,b\rangle \neq \emptyset$ and for every morphism f from a to b holds $\mathcal{C}(f) = \mathcal{C}(a,b,f)$,

- For all objects a, b of $\mathcal{A}$ such that $\mathcal{F}(a) = \mathcal{F}(b)$ holds $a = b$,
- For all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for all morphisms f, g from a to b such that $\mathcal{C}(a,b,f) = \mathcal{C}(a,b,g)$ holds $f = g$, and
- Let a, b be objects of $\mathcal{B}$. Suppose $\langle a, b\rangle \neq \emptyset$. Let f be a morphism from a to b. Then there exist objects c, d of $\mathcal{A}$ and there exists a morphism g from c to d such that $b = \mathcal{F}(c)$ and $a = \mathcal{F}(d)$ and $\langle c, d\rangle \neq \emptyset$ and $f = \mathcal{C}(c,d,g)$.

The scheme *CatAntiIsomorphism* deals with categories $\mathcal{A}$, $\mathcal{B}$, a unary functor $\mathcal{F}$ yielding a set, and a ternary functor $\mathcal{G}$ yielding a set, and states that:

$\mathcal{A}$, $\mathcal{B}$ are anti-isomorphic

provided the parameters meet the following conditions:

- There exists a contravariant functor F from $\mathcal{A}$ to $\mathcal{B}$ such that
  (i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(a)$, and
  (ii) for all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{G}(a,b,f)$,
- For all objects a, b of $\mathcal{A}$ such that $\mathcal{F}(a) = \mathcal{F}(b)$ holds $a = b$,
- For all objects a, b of $\mathcal{A}$ such that $\langle a, b\rangle \neq \emptyset$ and for all morphisms f, g from a to b such that $\mathcal{G}(a,b,f) = \mathcal{G}(a,b,g)$ holds $f = g$, and
- Let a, b be objects of $\mathcal{B}$. Suppose $\langle a, b\rangle \neq \emptyset$. Let f be a morphism from a to b. Then there exist objects c, d of $\mathcal{A}$ and there exists a morphism g from c to d such that $b = \mathcal{F}(c)$ and $a = \mathcal{F}(d)$ and $\langle c, d\rangle \neq \emptyset$ and $f = \mathcal{G}(c,d,g)$.

Let A, B be categories. We say that A and B are equivalent if and only if the condition (Def. 2) is satisfied.

(Def. 2) There exists a covariant functor F from A to B and there exists a covariant functor G from B to A such that $G \cdot F$ and id_A are naturally equivalent and $F \cdot G$ and id_B are naturally equivalent.

Let us notice that the predicate A and B are equivalent is reflexive and symmetric.

Next we state three propositions:

(3) Let A, B, C be non empty reflexive graphs, F_1, F_2 be feasible functor structures from A to B, and G_1, G_2 be functor structures from B to C. Suppose that

(i) the functor structure of $F_1 =$ the functor structure of F_2, and

(ii) the functor structure of $G_1 =$ the functor structure of G_2.

Then $G_1 \cdot F_1 = G_2 \cdot F_2$.

(4) Let A, B, C be categories. Suppose A and B are equivalent and B and C are equivalent. Then A and C are equivalent.

(5) For all categories A, B such that A and B are isomorphic holds A and B are equivalent.

Now we present two schemes. The scheme *NatTransLambda* deals with categories $\mathcal{A}$, $\mathcal{B}$, covariant functors $\mathcal{C}$, $\mathcal{D}$ from $\mathcal{A}$ to $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a natural transformation t from $\mathcal{C}$ to $\mathcal{D}$ such that $\mathcal{C}$ is naturally transformable to $\mathcal{D}$ and for every object a of $\mathcal{A}$ holds $t[a] = \mathcal{F}(a)$

provided the parameters meet the following requirements:

- For every object a of $\mathcal{A}$ holds $\mathcal{F}(a) \in \langle \mathcal{C}(a), \mathcal{D}(a)\rangle$, and
- Let a, b be objects of $\mathcal{A}$. Suppose $\langle a, b\rangle \neq \emptyset$. Let f be a morphism from a to b and g_1 be a morphism from $\mathcal{C}(a)$ to $\mathcal{D}(a)$. Suppose $g_1 = \mathcal{F}(a)$. Let g_2 be a morphism from $\mathcal{C}(b)$ to $\mathcal{D}(b)$. If $g_2 = \mathcal{F}(b)$, then $g_2 \cdot \mathcal{C}(f) = \mathcal{D}(f) \cdot g_1$.

The scheme *NatEquivalenceLambda* deals with categories $\mathcal{A}$, $\mathcal{B}$, covariant functors $\mathcal{C}$, $\mathcal{D}$ from $\mathcal{A}$ to $\mathcal{B}$, and a unary functor $\mathcal{F}$ yielding a set, and states that:

There exists a natural equivalence t of $\mathcal{C}$ and $\mathcal{D}$ such that $\mathcal{C}$ and $\mathcal{D}$ are naturally equivalent and for every object a of $\mathcal{A}$ holds $t[a] = \mathcal{F}(a)$

provided the parameters meet the following requirements:

- Let a be an object of $\mathcal{A}$. Then $\mathcal{F}(a) \in \langle \mathcal{C}(a), \mathcal{D}(a)\rangle$ and $\langle \mathcal{D}(a), \mathcal{C}(a)\rangle \neq \emptyset$ and for every morphism f from $\mathcal{C}(a)$ to $\mathcal{D}(a)$ such that $f = \mathcal{F}(a)$ holds f is iso, and

- Let a, b be objects of $\mathcal{A}$. Suppose $\langle a,b\rangle \neq \emptyset$. Let f be a morphism from a to b and g_1 be a morphism from $\mathcal{C}(a)$ to $\mathcal{D}(a)$. Suppose $g_1 = \mathcal{F}(a)$. Let g_2 be a morphism from $\mathcal{C}(b)$ to $\mathcal{D}(b)$. If $g_2 = \mathcal{F}(b)$, then $g_2 \cdot \mathcal{C}(f) = \mathcal{D}(f) \cdot g_1$.

3. Dual Categories

Let C_1, C_2 be non empty category structures. We say that C_1 and C_2 are opposite if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) The carrier of C_2 = the carrier of C_1,

(ii) the arrows of C_2 = $\curvearrowleft$(the arrows of C_1), and

(iii) for all objects a, b, c of C_1 and for all objects a', b', c' of C_2 such that $a' = a$ and $b' = b$ and $c' = c$ holds (the composition of C_2)(a', b', c') = $\curvearrowleft$(the composition of C_1)(c, b, a).

Let us note that the predicate C_1 and C_2 are opposite is symmetric.

Next we state several propositions:

(6) For all non empty category structures A, B such that A and B are opposite holds every object of A is an object of B.

(7) Let A, B be non empty category structures. Suppose A and B are opposite. Let a, b be objects of A and a', b' be objects of B. If $a' = a$ and $b' = b$, then $\langle a,b\rangle = \langle b',a'\rangle$.

(8) Let A, B be non empty category structures. Suppose A and B are opposite. Let a, b be objects of A and a', b' be objects of B. If $a' = a$ and $b' = b$, then every morphism from a to b is a morphism from b' to a'.

(9) Let C_1, C_2 be non empty transitive category structures. Then C_1 and C_2 are opposite if and only if the following conditions are satisfied:

(i) the carrier of C_2 = the carrier of C_1, and

(ii) for all objects a, b, c of C_1 and for all objects a', b', c' of C_2 such that $a' = a$ and $b' = b$ and $c' = c$ holds $\langle a,b\rangle = \langle b',a'\rangle$ and if $\langle a,b\rangle \neq \emptyset$ and $\langle b,c\rangle \neq \emptyset$, then for every morphism f from a to b and for every morphism g from b to c and for every morphism f' from b' to a' and for every morphism g' from c' to b' such that $f' = f$ and $g' = g$ holds $f' \cdot g' = g \cdot f$.

(10) Let A, B be categories. Suppose A and B are opposite. Let a be an object of A and b be an object of B. If $a = b$, then $\mathrm{id}_a = \mathrm{id}_b$.

(11) Let C be a transitive non empty category structure. Then there exists a strict transitive non empty category structure C' such that C and C' are opposite.

(12) Let A, B be transitive non empty category structures. Suppose A and B are opposite. If A is associative, then B is associative.

(13) For all transitive non empty category structures A, B such that A and B are opposite holds if A has units, then B has units.

(14) Let C, C_1, C_2 be non empty category structures. Suppose C and C_1 are opposite. Then C and C_2 are opposite if and only if the category structure of C_1 = the category structure of C_2.

Let C be a transitive non empty category structure. The functor C^{op} yielding a strict transitive non empty category structure is defined as follows:

(Def. 4) C and C^{op} are opposite.

Let C be an associative transitive non empty category structure. One can verify that C^{op} is associative.

Let C be a transitive non empty category structure with units. Note that C^{op} has units.

Let A, B be categories. Let us assume that A and B are opposite. The dualizing functor from A into B is a contravariant strict functor from A to B and is defined by the conditions (Def. 5).

(Def. 5)(i) For every object a of A holds (the dualizing functor from A into $B)(a) = a$, and

(ii) for all objects a, b of A such that $\langle a,b\rangle \neq \emptyset$ and for every morphism f from a to b holds (the dualizing functor from A into $B)(f) = f$.

Next we state two propositions:

(15) Let A, B be categories. Suppose A and B are opposite. Then (the dualizing functor from A into $B) \cdot$ (the dualizing functor from B into $A) = \mathrm{id}_B$.

(16) Let A, B be categories. Suppose A and B are opposite. Then the dualizing functor from A into B is bijective.

Let A be a category. Observe that the dualizing functor from A into A^{op} is bijective and the dualizing functor from A^{op} into A is bijective.

Next we state a number of propositions:

(17) For all categories A, B such that A and B are opposite holds A, B are anti-isomorphic.

(18) Let A, B, C be categories. Suppose A and B are opposite. Then A and C are isomorphic if and only if B, C are anti-isomorphic.

(19) Let A, B, C, D be categories. Suppose A and B are opposite and C and D are opposite. If A and C are isomorphic, then B and D are isomorphic.

(20) Let A, B, C, D be categories. Suppose A and B are opposite and C and D are opposite. If A, C are anti-isomorphic, then B, D are anti-isomorphic.

(21) Let A, B be categories. Suppose A and B are opposite. Let a, b be objects of A. Suppose $\langle a,b\rangle \neq \emptyset$ and $\langle b,a\rangle \neq \emptyset$. Let a', b' be objects of B. Suppose $a' = a$ and $b' = b$. Let f be a morphism from a to b and f' be a morphism from b' to a'. If $f' = f$, then f is retraction iff f' is coretraction.

(22) Let A, B be categories. Suppose A and B are opposite. Let a, b be objects of A. Suppose $\langle a,b\rangle \neq \emptyset$ and $\langle b,a\rangle \neq \emptyset$. Let a', b' be objects of B. Suppose $a' = a$ and $b' = b$. Let f be a morphism from a to b and f' be a morphism from b' to a'. If $f' = f$, then f is coretraction iff f' is retraction.

(23) Let A, B be categories. Suppose A and B are opposite. Let a, b be objects of A. Suppose $\langle a,b\rangle \neq \emptyset$ and $\langle b,a\rangle \neq \emptyset$. Let a', b' be objects of B. Suppose $a' = a$ and $b' = b$. Let f be a morphism from a to b and f' be a morphism from b' to a'. If $f' = f$ and f is retraction and coretraction, then $f'^{-1} = f^{-1}$.

(24) Let A, B be categories. Suppose A and B are opposite. Let a, b be objects of A. Suppose $\langle a,b\rangle \neq \emptyset$ and $\langle b,a\rangle \neq \emptyset$. Let a', b' be objects of B. Suppose $a' = a$ and $b' = b$. Let f be a morphism from a to b and f' be a morphism from b' to a'. If $f' = f$, then f is iso iff f' is iso.

(25) Let A, B, C, D be categories. Suppose A and B are opposite and C and D are opposite. Let F, G be covariant functors from B to C. Suppose F and G are naturally equivalent. Then (the dualizing functor from C into $D) \cdot G \cdot$ the dualizing functor from A into B and (the dualizing functor from C into $D) \cdot F \cdot$ the dualizing functor from A into B are naturally equivalent.

(26) Let A, B, C, D be categories. Suppose A and B are opposite and C and D are opposite. If A and C are equivalent, then B and D are equivalent.

Let A, B be categories. We say that A and B are dual if and only if:

(Def. 6) A and B^{op} are equivalent.

Let us note that the predicate A and B are dual is symmetric.

The following four propositions are true:

(27) For all categories A, B such that A, B are anti-isomorphic holds A and B are dual.

(28) Let A, B, C be categories. Suppose A and B are opposite. Then A and C are equivalent if and only if B and C are dual.

(29) For all categories A, B, C such that A and B are dual and B and C are equivalent holds A and C are dual.

(30) For all categories A, B, C such that A and B are dual and B and C are dual holds A and C are equivalent.

4. Concrete Categories

The following proposition is true

(31) For all sets X, Y, x holds $x \in Y^X$ iff x is a function and $\pi_1(x) = X$ and $\pi_2(x) \subseteq Y$.

Let C be a 1-sorted structure. A many sorted set indexed by C is a many sorted set indexed by the carrier of C.

Let C be a category. We say that C is para-functional if and only if:

(Def. 7) There exists a many sorted set F indexed by C such that for all objects a_1, a_2 of C holds $\langle a_1, a_2 \rangle \subseteq F(a_2)^{F(a_1)}$.

Let us note that every category which is quasi-functional is also para-functional.

Let C be a category and let a be a set. C-carrier of a is defined by:

(Def. 8)(i) There exists an object b of C such that $b = a$ and C-carrier of $a = \pi_1(\mathrm{id}_b)$ if a is an object of C,

(ii) C-carrier of $a = \emptyset$, otherwise.

Let C be a category and let a be an object of C. Then C-carrier of a can be characterized by the condition:

(Def. 9) C-carrier of $a = \pi_1(\mathrm{id}_a)$.

We introduce the carrier of a as a synonym of C-carrier of a.

We now state two propositions:

(32) For every non empty set A and for every object a of Ens_A holds $\mathrm{id}_a = \mathrm{id}_a$.

(33) For every non empty set A and for every object a of Ens_A holds the carrier of $a = a$.

Let C be a category. We say that C is set-id-inheriting if and only if:

(Def. 10) For every object a of C holds $\mathrm{id}_a = \mathrm{id}_{\text{the carrier of } a}$.

Let A be a non empty set. Note that Ens_A is set-id-inheriting.

Let C be a category. We say that C is concrete if and only if:

(Def. 11) C is para-functional, semi-functional, and set-id-inheriting.

Let us note that every category which is concrete is also para-functional, semi-functional, and set-id-inheriting and every category which is para-functional, semi-functional, and set-id-inheriting is also concrete.

Let us observe that there exists a category which is concrete, quasi-functional, and strict.

We now state two propositions:

(34) Let C be a category. Then C is para-functional if and only if for all objects a_1, a_2 of C holds $\langle a_1, a_2 \rangle \subseteq (\text{the carrier of } a_2)^{\text{the carrier of } a_1}$.

(35) Let C be a para-functional category and a, b be objects of C. Suppose $\langle a,b\rangle \neq \emptyset$. Then every morphism from a to b is a function from the carrier of a into the carrier of b.

Let A be a para-functional category and let a, b be objects of A. Observe that every morphism from a to b is function-like and relation-like.

We now state four propositions:

(36) Let C be a para-functional category and a, b be objects of C. Suppose $\langle a,b\rangle \neq \emptyset$. Let f be a morphism from a to b. Then $\operatorname{dom} f =$ the carrier of a and $\operatorname{rng} f \subseteq$ the carrier of b.

(37) For every para-functional semi-functional category C and for every object a of C holds the carrier of $a = \operatorname{dom}(\mathrm{id}_a)$.

(38) Let C be a para-functional semi-functional category and a, b, c be objects of C. Suppose $\langle a,b\rangle \neq \emptyset$ and $\langle b,c\rangle \neq \emptyset$. Let f be a morphism from a to b and g be a morphism from b to c. Then $g \cdot f = (g$ **qua** function$) \cdot (f$ **qua** function$)$.

(39) Let C be a para-functional semi-functional category and a be an object of C. If $\mathrm{id}_{\text{the carrier of } a} \in \langle a,a\rangle$, then $\mathrm{id}_a = \mathrm{id}_{\text{the carrier of } a}$.

Now we present several schemes. The scheme *ConcreteCategoryLambda* deals with a non empty set $\mathcal{A}$, a binary functor $\mathcal{F}$ yielding a set, and a unary functor $\mathcal{G}$ yielding a set, and states that:

There exists a concrete strict category C such that
- (i) the carrier of $C = \mathcal{A}$,
- (ii) for every object a of C holds the carrier of $a = \mathcal{G}(a)$, and
- (iii) for all objects a, b of C holds $\langle a,b\rangle = \mathcal{F}(a,b)$

provided the parameters have the following properties:
- For all elements a, b, c of $\mathcal{A}$ and for all functions f, g such that $f \in \mathcal{F}(a,b)$ and $g \in \mathcal{F}(b,c)$ holds $g \cdot f \in \mathcal{F}(a,c)$,
- For all elements a, b of $\mathcal{A}$ holds $\mathcal{F}(a,b) \subseteq \mathcal{G}(b)^{\mathcal{G}(a)}$, and
- For every element a of $\mathcal{A}$ holds $\mathrm{id}_{\mathcal{G}(a)} \in \mathcal{F}(a,a)$.

The scheme *ConcreteCategoryQuasiLambda* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and a ternary predicate $\mathcal{P}$, and states that:

There exists a concrete strict category C such that
- (i) the carrier of $C = \mathcal{A}$,
- (ii) for every object a of C holds the carrier of $a = \mathcal{F}(a)$, and
- (iii) for all elements a, b of $\mathcal{A}$ and for every function f holds $f \in$ (the arrows of $C)(a, b)$ iff $f \in \mathcal{F}(b)^{\mathcal{F}(a)}$ and $\mathcal{P}[a,b,f]$

provided the parameters meet the following requirements:
- For all elements a, b, c of $\mathcal{A}$ and for all functions f, g such that $\mathcal{P}[a,b,f]$ and $\mathcal{P}[b,c,g]$ holds $\mathcal{P}[a,c,g\cdot f]$, and
- For every element a of $\mathcal{A}$ holds $\mathcal{P}[a,a,\mathrm{id}_{\mathcal{F}(a)}]$.

The scheme *ConcreteCategoryEx* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, a binary predicate $\mathcal{P}$, and a ternary predicate $\mathcal{Q}$, and states that:

There exists a concrete strict category C such that
- (i) the carrier of $C = \mathcal{A}$,
- (ii) for every object a of C and for every set x holds $x \in$ the carrier of a iff $x \in \mathcal{F}(a)$ and $\mathcal{P}[a,x]$, and
- (iii) for all elements a, b of $\mathcal{A}$ and for every function f holds $f \in$ (the arrows of $C)(a, b)$ iff $f \in ($ C-carrier of $b)^{C\text{-carrier of } a}$ and $\mathcal{Q}[a,b,f]$

provided the parameters meet the following requirements:
- For all elements a, b, c of $\mathcal{A}$ and for all functions f, g such that $\mathcal{Q}[a,b,f]$ and $\mathcal{Q}[b,c,g]$ holds $\mathcal{Q}[a,c,g\cdot f]$, and
- Let a be an element of $\mathcal{A}$ and X be a set. If for every set x holds $x \in X$ iff $x \in \mathcal{F}(a)$ and $\mathcal{P}[a,x]$, then $\mathcal{Q}[a,a,\mathrm{id}_X]$.

The scheme *ConcreteCategoryUniq1* deals with a non empty set $\mathcal{A}$ and a binary functor $\mathcal{F}$ yielding a set, and states that:

Let C_1, C_2 be para-functional semi-functional categories. Suppose that
(i) the carrier of $C_1 = \mathcal{A}$,
(ii) for all objects a, b of C_1 holds $\langle a,b\rangle = \mathcal{F}(a,b)$,
(iii) the carrier of $C_2 = \mathcal{A}$, and
(iv) for all objects a, b of C_2 holds $\langle a,b\rangle = \mathcal{F}(a,b)$.
Then the category structure of $C_1 =$ the category structure of C_2
for all values of the parameters.

The scheme *ConcreteCategoryUniq2* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, and a ternary predicate $\mathcal{P}$, and states that:

Let C_1, C_2 be para-functional semi-functional categories. Suppose that
(i) the carrier of $C_1 = \mathcal{A}$,
(ii) for all elements a, b of $\mathcal{A}$ and for every function f holds $f \in$ (the arrows of $C_1)(a, b)$ iff $f \in \mathcal{F}(b)^{\mathcal{F}(a)}$ and $\mathcal{P}[a,b,f]$,
(iii) the carrier of $C_2 = \mathcal{A}$, and
(iv) for all elements a, b of $\mathcal{A}$ and for every function f holds $f \in$ (the arrows of $C_2)(a, b)$ iff $f \in \mathcal{F}(b)^{\mathcal{F}(a)}$ and $\mathcal{P}[a,b,f]$.
Then the category structure of $C_1 =$ the category structure of C_2
for all values of the parameters.

The scheme *ConcreteCategoryUniq3* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, a binary predicate $\mathcal{P}$, and a ternary predicate $\mathcal{Q}$, and states that:

Let C_1, C_2 be para-functional semi-functional categories. Suppose that
(i) the carrier of $C_1 = \mathcal{A}$,
(ii) for every object a of C_1 and for every set x holds $x \in$ the carrier of a iff $x \in \mathcal{F}(a)$ and $\mathcal{P}[a,x]$,
(iii) for all elements a, b of $\mathcal{A}$ and for every function f holds $f \in$ (the arrows of $C_1)(a, b)$ iff $f \in (C_1\text{-carrier of } b)^{C_1\text{-carrier of } a}$ and $\mathcal{Q}[a,b,f]$,
(iv) the carrier of $C_2 = \mathcal{A}$,
(v) for every object a of C_2 and for every set x holds $x \in$ the carrier of a iff $x \in \mathcal{F}(a)$ and $\mathcal{P}[a,x]$, and
(vi) for all elements a, b of $\mathcal{A}$ and for every function f holds $f \in$ (the arrows of $C_2)(a, b)$ iff $f \in (C_2\text{-carrier of } b)^{C_2\text{-carrier of } a}$ and $\mathcal{Q}[a,b,f]$.
Then the category structure of $C_1 =$ the category structure of C_2
for all values of the parameters.

5. Equivalence Between Concrete Categories

We now state several propositions:

(40) Let C be a concrete category and a, b be objects of C. Suppose $\langle a,b\rangle \neq \emptyset$ and $\langle b,a\rangle \neq \emptyset$. Let f be a morphism from a to b. If f is retraction, then $\operatorname{rng} f =$ the carrier of b.

(41) Let C be a concrete category and a, b be objects of C. Suppose $\langle a,b\rangle \neq \emptyset$ and $\langle b,a\rangle \neq \emptyset$. Let f be a morphism from a to b. If f is coretraction, then f is one-to-one.

(42) Let C be a concrete category and a, b be objects of C. Suppose $\langle a,b\rangle \neq \emptyset$ and $\langle b,a\rangle \neq \emptyset$. Let f be a morphism from a to b. If f is iso, then f is one-to-one and $\operatorname{rng} f =$ the carrier of b.

(43) Let C be a para-functional semi-functional category and a, b be objects of C. Suppose $\langle a,b\rangle \neq \emptyset$. Let f be a morphism from a to b. If f is one-to-one and (f **qua** function) $^{-1} \in \langle b,a\rangle$, then f is iso.

(44) Let C be a concrete category and a, b be objects of C. Suppose $\langle a,b\rangle \neq \emptyset$ and $\langle b,a\rangle \neq \emptyset$. Let f be a morphism from a to b. If f is iso, then $f^{-1} =$ (f **qua** function) $^{-1}$.

The scheme *ConcreteCatEquivalence* deals with para-functional semi-functional categories $\mathcal{A}$, $\mathcal{B}$, two unary functors $\mathcal{F}$ and $\mathcal{G}$ yielding sets, two ternary functors $\mathcal{H}$ and $\mathcal{I}$ yielding functions, and two unary functors $\mathcal{A}$ and $\mathcal{B}$ yielding functions, and states that:

$\mathcal{A}$ and $\mathcal{B}$ are equivalent

provided the parameters meet the following requirements:

- There exists a covariant functor F from $\mathcal{A}$ to $\mathcal{B}$ such that
  (i) for every object a of $\mathcal{A}$ holds $F(a) = \mathcal{F}(a)$, and
  (ii) for all objects a, b of $\mathcal{A}$ such that $\langle a,b\rangle \neq \emptyset$ and for every morphism f from a to b holds $F(f) = \mathcal{H}(a,b,f)$,
- There exists a covariant functor G from $\mathcal{B}$ to $\mathcal{A}$ such that
  (i) for every object a of $\mathcal{B}$ holds $G(a) = \mathcal{G}(a)$, and
  (ii) for all objects a, b of $\mathcal{B}$ such that $\langle a,b\rangle \neq \emptyset$ and for every morphism f from a to b holds $G(f) = \mathcal{I}(a,b,f)$,
- For all objects a, b of $\mathcal{A}$ such that $a = \mathcal{G}(\mathcal{F}(b))$ holds $\mathcal{A}(b) \in \langle a,b\rangle$ and $\mathcal{A}(b)^{-1} \in \langle b,a\rangle$ and $\mathcal{A}(b)$ is one-to-one,
- For all objects a, b of $\mathcal{B}$ such that $b = \mathcal{F}(\mathcal{G}(a))$ holds $\mathcal{B}(a) \in \langle a,b\rangle$ and $\mathcal{B}(a)^{-1} \in \langle b,a\rangle$ and $\mathcal{B}(a)$ is one-to-one,
- For all objects a, b of $\mathcal{A}$ such that $\langle a,b\rangle \neq \emptyset$ and for every morphism f from a to b holds $\mathcal{A}(b) \cdot \mathcal{I}(\mathcal{F}(a), \mathcal{F}(b), \mathcal{H}(a,b,f)) = f \cdot \mathcal{A}(a)$, and
- For all objects a, b of $\mathcal{B}$ such that $\langle a,b\rangle \neq \emptyset$ and for every morphism f from a to b holds $\mathcal{H}(\mathcal{G}(a), \mathcal{G}(b), \mathcal{I}(a,b,f)) \cdot \mathcal{B}(a) = \mathcal{B}(b) \cdot f$.

6. Concretization of Categories

Let C be a category. The concretized C is a concrete strict category and is defined by the conditions (Def. 12).

(Def. 12)(i) The carrier of the concretized $C =$ the carrier of C,

(ii) for every object a of the concretized C and for every set x holds $x \in$ the carrier of a iff $x \in \bigcup$disjoint(the arrows of C) and $a = x_{\mathbf{2},\mathbf{2}}$, and

(iii) for all elements a, b of C and for every function f holds $f \in$ (the arrows of the concretized C)(a, b) iff $f \in$ ((the concretized C)-carrier of b)$^{(\text{the concretized } C)\text{-carrier of } a}$ and there exist objects f_1, f_2 of C and there exists a morphism g from f_1 to f_2 such that $f_1 = a$ and $f_2 = b$ and $\langle f_1, f_2\rangle \neq \emptyset$ and for every object o of C such that $\langle o, f_1\rangle \neq \emptyset$ and for every morphism h from o to f_1 holds $f(\langle h, \langle o, f_1\rangle\rangle) = \langle g \cdot h, \langle o, f_2\rangle\rangle$.

Next we state the proposition

(45) Let A be a category, a be an object of A, and x be a set. Then $x \in$ (the concretized A)-carrier of a if and only if there exists an object b of A and there exists a morphism f from b to a such that $\langle b,a\rangle \neq \emptyset$ and $x = \langle f, \langle b, a\rangle\rangle$.

Let A be a category and let a be an object of A. Observe that (the concretized A)-carrier of a is non empty.

Next we state two propositions:

(46) Let A be a category and a, b be objects of A. Suppose $\langle a,b\rangle \neq \emptyset$. Let f be a morphism from a to b. Then there exists a function F from (the concretized A)-carrier of a into (the concretized A)-carrier of b such that

(i) $F \in$ (the arrows of the concretized A)(a, b), and

(ii) for every object c of A and for every morphism g from c to a such that $\langle c,a\rangle \neq \emptyset$ holds $F(\langle g, \langle c, a\rangle\rangle) = \langle f \cdot g, \langle c, b\rangle\rangle$.

(47) Let A be a category and a, b be objects of A. Suppose $\langle a,b\rangle \neq \emptyset$. Let F_1, F_2 be functions. Suppose that

(i) $F_1 \in$ (the arrows of the concretized A)(a, b),

(ii) $F_2 \in$ (the arrows of the concretized A)(a, b), and

(iii) $F_1(\langle \mathrm{id}_a, \langle a, a\rangle\rangle) = F_2(\langle \mathrm{id}_a, \langle a, a\rangle\rangle)$.

Then $F_1 = F_2$.

The scheme *NonUniqMSFunctionEx* deals with a set $\mathcal{A}$, many sorted sets $\mathcal{B}$, $\mathcal{C}$ indexed by $\mathcal{A}$, and a ternary predicate $\mathcal{P}$, and states that:

> There exists a many sorted function F from $\mathcal{B}$ into $\mathcal{C}$ such that for all sets i, x if $i \in \mathcal{A}$ and $x \in \mathcal{B}(i)$, then $F(i)(x) \in \mathcal{C}(i)$ and $\mathcal{P}[i,x,F(i)(x)]$

provided the parameters meet the following condition:

- For all sets i, x such that $i \in \mathcal{A}$ and $x \in \mathcal{B}(i)$ there exists a set y such that $y \in \mathcal{C}(i)$ and $\mathcal{P}[i,x,y]$.

Let A be a category. The concretization of A is a covariant strict functor from A to the concretized A and is defined by the conditions (Def. 13).

(Def. 13)(i) For every object a of A holds (the concretization of $A)(a) = a$, and

(ii) for all objects a, b of A such that $\langle a,b\rangle \neq \emptyset$ and for every morphism f from a to b holds (the concretization of $A)(f)(\langle \mathrm{id}_a, \langle a, a\rangle\rangle) = \langle f, \langle a, b\rangle\rangle$.

Let A be a category. One can verify that the concretization of A is bijective.

The following proposition is true

(48) For every category A holds A and the concretized A are isomorphic.

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