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On the Characterization of Modular and Distributive Lattices[1]

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Summary. This article contains definitions of the "pentagon" lattice N_5 and the "diamond" lattice M_3. It is followed by the characterization of modular and distributive lattices depending on the possible shape of substructures. The last part treats of interval-like sublattices of any lattice.

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The articles [11], [10], [13], [14], [5], [6], [12], [2], [3], [9], [15], [7], [8], [4], and [1] provide the notation and terminology for this paper.

1. Preliminaries

The following propositions are true:

(1) $3 = \{0,1,2\}$.

(2) $2 \setminus 1 = \{1\}$.

(3) $3 \setminus 1 = \{1,2\}$.

(4) $3 \setminus 2 = \{2\}$.

2. Main Part

One can prove the following propositions:

(5) Let L be an antisymmetric reflexive relational structure with g.l.b.'s and l.u.b.'s and a, b be elements of L. Then $a \sqcap b = b$ if and only if $a \sqcup b = a$.

(6) For every lattice L and for all elements a, b, c of L holds $(a \sqcap b) \sqcup (a \sqcap c) \leq a \sqcap (b \sqcup c)$.

(7) For every lattice L and for all elements a, b, c of L holds $a \sqcup (b \sqcap c) \leq (a \sqcup b) \sqcap (a \sqcup c)$.

(8) For every lattice L and for all elements a, b, c of L such that $a \leq c$ holds $a \sqcup (b \sqcap c) \leq (a \sqcup b) \sqcap c$.

The relational structure N_5 is defined as follows:

(Def. 1) $N_5 = \langle \{0, 3 \setminus 1, 2, 3 \setminus 2, 3\}, \subseteq \rangle$.

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Let us note that N_5 is strict, reflexive, transitive, and antisymmetric and N_5 has g.l.b.'s and l.u.b.'s.

The relational structure M_3 is defined by:

(Def. 2) $M_3 = \langle \{0,1,2 \setminus 1,3 \setminus 2,3\}, \subseteq \rangle$.

Let us note that M_3 is strict, reflexive, transitive, and antisymmetric and M_3 has g.l.b.'s and l.u.b.'s.

One can prove the following propositions:

(9) Let L be a lattice. Then the following statements are equivalent

(i) there exists a full sublattice K of L such that N_5 and K are isomorphic,

(ii) there exist elements a, b, c, d, e of L such that $a \neq b$ and $a \neq c$ and $a \neq d$ and $a \neq e$ and $b \neq c$ and $b \neq d$ and $b \neq e$ and $c \neq d$ and $c \neq e$ and $d \neq e$ and $a \sqcap b = a$ and $a \sqcap c = a$ and $c \sqcap e = c$ and $d \sqcap e = d$ and $b \sqcap c = a$ and $b \sqcap d = b$ and $c \sqcap d = a$ and $b \sqcup c = e$ and $c \sqcup d = e$.

(10) Let L be a lattice. Then the following statements are equivalent

(i) there exists a full sublattice K of L such that M_3 and K are isomorphic,

(ii) there exist elements a, b, c, d, e of L such that $a \neq b$ and $a \neq c$ and $a \neq d$ and $a \neq e$ and $b \neq c$ and $b \neq d$ and $b \neq e$ and $c \neq d$ and $c \neq e$ and $d \neq e$ and $a \sqcap b = a$ and $a \sqcap c = a$ and $a \sqcap d = a$ and $b \sqcap e = b$ and $c \sqcap e = c$ and $d \sqcap e = d$ and $b \sqcap c = a$ and $b \sqcap d = a$ and $c \sqcap d = a$ and $b \sqcup c = e$ and $b \sqcup d = e$ and $c \sqcup d = e$.

3. Diamond and Pentagon

Let L be a non empty relational structure. We say that L is modular if and only if:

(Def. 3) For all elements a, b, c of L such that $a \leq c$ holds $a \sqcup (b \sqcap c) = (a \sqcup b) \sqcap c$.

One can check that every non empty antisymmetric reflexive relational structure with g.l.b.'s which is distributive is also modular.

The following two propositions are true:

(11) Let L be a lattice. Then L is modular if and only if it is not true that there exists a full sublattice K of L such that N_5 and K are isomorphic.

(12) Let L be a lattice. Suppose L is modular. Then L is distributive if and only if it is not true that there exists a full sublattice K of L such that M_3 and K are isomorphic.

4. Intervals of a Lattice

Let L be a non empty relational structure and let a, b be elements of L. The functor $[a,b]$ yields a subset of L and is defined by:

(Def. 4) For every element c of L holds $c \in [a,b]$ iff $a \leq c$ and $c \leq b$.

Let L be a non empty relational structure and let I_1 be a subset of L. We say that I_1 is interval if and only if:

(Def. 5) There exist elements a, b of L such that $I_1 = [a,b]$.

Let L be a non empty reflexive transitive relational structure. One can check that every subset of L which is non empty and interval is also directed and every subset of L which is non empty and interval is also filtered.

Let L be a non empty relational structure and let a, b be elements of L. One can check that $[a,b]$ is interval.

The following proposition is true

(13) For every non empty reflexive transitive relational structure L and for all elements a, b of L holds $[a,b] = \uparrow a \cap \downarrow b$.

Let L be a poset with g.l.b.'s and let a, b be elements of L. One can check that $\mathrm{sub}([a,b])$ is meet-inheriting.

Let L be a poset with l.u.b.'s and let a, b be elements of L. One can check that $\mathrm{sub}([a,b])$ is join-inheriting.

We now state the proposition

(14) Let L be a lattice and a, b be elements of L. If L is modular, then $\mathrm{sub}([b, a \sqcup b])$ and $\mathrm{sub}([a \sqcap b, a])$ are isomorphic.

Let us note that there exists a lattice which is finite and non empty.

Let us note that every semilattice which is finite is also lower-bounded.

Let us note that every lattice which is finite is also complete.

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