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The Properties of Product of Relational Structures[1]

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Summary. This work contains useful facts about the product of relational structures. It continues the formalization of [7].

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The articles [13], [5], [14], [16], [1], [2], [3], [6], [8], [9], [10], [4], [11], [12], and [15] provide the notation and terminology for this paper.

1. On the Elements of Product of Relational Structures

Let S, T be non empty upper-bounded relational structures. Observe that $[:S,T:]$ is upper-bounded.

Let S, T be non empty lower-bounded relational structures. Observe that $[:S,T:]$ is lower-bounded.

One can prove the following propositions:

(1) Let S, T be non empty relational structures. If $[:S,T:]$ is upper-bounded, then S is upper-bounded and T is upper-bounded.

(2) Let S, T be non empty relational structures. If $[:S,T:]$ is lower-bounded, then S is lower-bounded and T is lower-bounded.

(3) For all upper-bounded antisymmetric non empty relational structures S, T holds $\top_{[:S,T:]} = \langle \top_S, \top_T \rangle$.

(4) For all lower-bounded antisymmetric non empty relational structures S, T holds $\bot_{[:S,T:]} = \langle \bot_S, \bot_T \rangle$.

(5) Let S, T be lower-bounded antisymmetric non empty relational structures and D be a subset of $[:S,T:]$. If $[:S,T:]$ is complete or sup D exists in $[:S,T:]$, then $\sup D = \langle \sup \pi_1(D), \sup \pi_2(D) \rangle$.

(6) Let S, T be upper-bounded antisymmetric non empty relational structures and D be a subset of $[:S,T:]$. If $[:S,T:]$ is complete or inf D exists in $[:S,T:]$, then $\inf D = \langle \inf \pi_1(D), \inf \pi_2(D) \rangle$.

(7) Let S, T be non empty relational structures and x, y be elements of $[:S,T:]$. Then $x \leq \{y\}$ if and only if the following conditions are satisfied:

(i) $x_1 \leq \{y_1\}$, and

(ii) $x_2 \leq \{y_2\}$.

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(8) Let S, T be non empty relational structures and x, y, z be elements of $[:S, T:]$. Then $x \leq \{y, z\}$ if and only if the following conditions are satisfied:

(i) $x_{\mathbf{1}} \leq \{y_{\mathbf{1}}, z_{\mathbf{1}}\}$, and

(ii) $x_{\mathbf{2}} \leq \{y_{\mathbf{2}}, z_{\mathbf{2}}\}$.

(9) Let S, T be non empty relational structures and x, y be elements of $[:S, T:]$. Then $x \geq \{y\}$ if and only if the following conditions are satisfied:

(i) $x_{\mathbf{1}} \geq \{y_{\mathbf{1}}\}$, and

(ii) $x_{\mathbf{2}} \geq \{y_{\mathbf{2}}\}$.

(10) Let S, T be non empty relational structures and x, y, z be elements of $[:S, T:]$. Then $x \geq \{y, z\}$ if and only if the following conditions are satisfied:

(i) $x_{\mathbf{1}} \geq \{y_{\mathbf{1}}, z_{\mathbf{1}}\}$, and

(ii) $x_{\mathbf{2}} \geq \{y_{\mathbf{2}}, z_{\mathbf{2}}\}$.

(11) Let S, T be non empty antisymmetric relational structures and x, y be elements of $[:S, T:]$. Then $\inf \{x, y\}$ exists in $[:S, T:]$ if and only if $\inf \{x_{\mathbf{1}}, y_{\mathbf{1}}\}$ exists in S and $\inf \{x_{\mathbf{2}}, y_{\mathbf{2}}\}$ exists in T.

(12) Let S, T be non empty antisymmetric relational structures and x, y be elements of $[:S, T:]$. Then $\sup \{x, y\}$ exists in $[:S, T:]$ if and only if $\sup \{x_{\mathbf{1}}, y_{\mathbf{1}}\}$ exists in S and $\sup \{x_{\mathbf{2}}, y_{\mathbf{2}}\}$ exists in T.

(13) Let S, T be antisymmetric relational structures with g.l.b.'s and x, y be elements of $[:S, T:]$. Then $(x \sqcap y)_{\mathbf{1}} = x_{\mathbf{1}} \sqcap y_{\mathbf{1}}$ and $(x \sqcap y)_{\mathbf{2}} = x_{\mathbf{2}} \sqcap y_{\mathbf{2}}$.

(14) Let S, T be antisymmetric relational structures with l.u.b.'s and x, y be elements of $[:S, T:]$. Then $(x \sqcup y)_{\mathbf{1}} = x_{\mathbf{1}} \sqcup y_{\mathbf{1}}$ and $(x \sqcup y)_{\mathbf{2}} = x_{\mathbf{2}} \sqcup y_{\mathbf{2}}$.

(15) Let S, T be antisymmetric relational structures with g.l.b.'s, x_1, y_1 be elements of S, and x_2, y_2 be elements of T. Then $\langle x_1 \sqcap y_1, x_2 \sqcap y_2 \rangle = \langle x_1, x_2 \rangle \sqcap \langle y_1, y_2 \rangle$.

(16) Let S, T be antisymmetric relational structures with l.u.b.'s, x_1, y_1 be elements of S, and x_2, y_2 be elements of T. Then $\langle x_1 \sqcup y_1, x_2 \sqcup y_2 \rangle = \langle x_1, x_2 \rangle \sqcup \langle y_1, y_2 \rangle$.

Let S be an antisymmetric relational structure with l.u.b.'s and g.l.b.'s and let x, y be elements of S. Let us note that the predicate y is a complement of x is symmetric.

Next we state several propositions:

(17) Let S, T be bounded antisymmetric relational structures with l.u.b.'s and g.l.b.'s and x, y be elements of $[:S, T:]$. Then x is a complement of y if and only if $x_{\mathbf{1}}$ is a complement of $y_{\mathbf{1}}$ and $x_{\mathbf{2}}$ is a complement of $y_{\mathbf{2}}$.

(18) Let S, T be antisymmetric up-complete non empty reflexive relational structures, a, c be elements of S, and b, d be elements of T. If $\langle a, b \rangle \ll \langle c, d \rangle$, then $a \ll c$ and $b \ll d$.

(19) Let S, T be up-complete non empty posets, a, c be elements of S, and b, d be elements of T. Then $\langle a, b \rangle \ll \langle c, d \rangle$ if and only if $a \ll c$ and $b \ll d$.

(20) Let S, T be antisymmetric up-complete non empty reflexive relational structures and x, y be elements of $[:S, T:]$. If $x \ll y$, then $x_{\mathbf{1}} \ll y_{\mathbf{1}}$ and $x_{\mathbf{2}} \ll y_{\mathbf{2}}$.

(21) Let S, T be up-complete non empty posets and x, y be elements of $[:S, T:]$. Then $x \ll y$ if and only if the following conditions are satisfied:

(i) $x_{\mathbf{1}} \ll y_{\mathbf{1}}$, and

(ii) $x_{\mathbf{2}} \ll y_{\mathbf{2}}$.

(22) Let S, T be antisymmetric up-complete non empty reflexive relational structures and x be an element of $[:S, T:]$. If x is compact, then $x_{\mathbf{1}}$ is compact and $x_{\mathbf{2}}$ is compact.

(23) Let S, T be up-complete non empty posets and x be an element of $[:S, T:]$. If $x_{\mathbf{1}}$ is compact and $x_{\mathbf{2}}$ is compact, then x is compact.

2. On the Subsets of Product of Relational Structures

Next we state a number of propositions:

(24) Let S, T be antisymmetric relational structures with g.l.b.'s and X, Y be subsets of $[:S, T:]$. Then $\pi_1(X \sqcap Y) = \pi_1(X) \sqcap \pi_1(Y)$ and $\pi_2(X \sqcap Y) = \pi_2(X) \sqcap \pi_2(Y)$.

(25) Let S, T be antisymmetric relational structures with l.u.b.'s and X, Y be subsets of $[:S, T:]$. Then $\pi_1(X \sqcup Y) = \pi_1(X) \sqcup \pi_1(Y)$ and $\pi_2(X \sqcup Y) = \pi_2(X) \sqcup \pi_2(Y)$.

(26) For all relational structures S, T and for every subset X of $[:S, T:]$ holds $\downarrow X \subseteq [:\downarrow\pi_1(X), \downarrow\pi_2(X):]$.

(27) For all relational structures S, T and for every subset X of S and for every subset Y of T holds $[:\downarrow X, \downarrow Y:] = \downarrow[:X, Y:]$.

(28) For all relational structures S, T and for every subset X of $[:S, T:]$ holds $\pi_1(\downarrow X) \subseteq \downarrow\pi_1(X)$ and $\pi_2(\downarrow X) \subseteq \downarrow\pi_2(X)$.

(29) Let S be a relational structure, T be a reflexive relational structure, and X be a subset of $[:S, T:]$. Then $\pi_1(\downarrow X) = \downarrow\pi_1(X)$.

(30) Let S be a reflexive relational structure, T be a relational structure, and X be a subset of $[:S, T:]$. Then $\pi_2(\downarrow X) = \downarrow\pi_2(X)$.

(31) For all relational structures S, T and for every subset X of $[:S, T:]$ holds $\uparrow X \subseteq [:\uparrow\pi_1(X), \uparrow\pi_2(X):]$.

(32) For all relational structures S, T and for every subset X of S and for every subset Y of T holds $[:\uparrow X, \uparrow Y:] = \uparrow[:X, Y:]$.

(33) For all relational structures S, T and for every subset X of $[:S, T:]$ holds $\pi_1(\uparrow X) \subseteq \uparrow\pi_1(X)$ and $\pi_2(\uparrow X) \subseteq \uparrow\pi_2(X)$.

(34) Let S be a relational structure, T be a reflexive relational structure, and X be a subset of $[:S, T:]$. Then $\pi_1(\uparrow X) = \uparrow\pi_1(X)$.

(35) Let S be a reflexive relational structure, T be a relational structure, and X be a subset of $[:S, T:]$. Then $\pi_2(\uparrow X) = \uparrow\pi_2(X)$.

(36) Let S, T be non empty relational structures, s be an element of S, and t be an element of T. Then $[:\downarrow s, \downarrow t:] = \downarrow\langle s, t\rangle$.

(37) For all non empty relational structures S, T and for every element x of $[:S, T:]$ holds $\pi_1(\downarrow x) \subseteq \downarrow(x_{\mathbf{1}})$ and $\pi_2(\downarrow x) \subseteq \downarrow(x_{\mathbf{2}})$.

(38) Let S be a non empty relational structure, T be a non empty reflexive relational structure, and x be an element of $[:S, T:]$. Then $\pi_1(\downarrow x) = \downarrow(x_{\mathbf{1}})$.

(39) Let S be a non empty reflexive relational structure, T be a non empty relational structure, and x be an element of $[:S, T:]$. Then $\pi_2(\downarrow x) = \downarrow(x_{\mathbf{2}})$.

(40) Let S, T be non empty relational structures, s be an element of S, and t be an element of T. Then $[:\uparrow s, \uparrow t:] = \uparrow\langle s, t\rangle$.

(41) For all non empty relational structures S, T and for every element x of $[:S, T:]$ holds $\pi_1(\uparrow x) \subseteq \uparrow(x_{\mathbf{1}})$ and $\pi_2(\uparrow x) \subseteq \uparrow(x_{\mathbf{2}})$.

(42) Let S be a non empty relational structure, T be a non empty reflexive relational structure, and x be an element of $[:S, T:]$. Then $\pi_1(\uparrow x) = \uparrow(x_{\mathbf{1}})$.

(43) Let S be a non empty reflexive relational structure, T be a non empty relational structure, and x be an element of $[: S, T :]$. Then $\pi_2(\uparrow x) = \uparrow(x_{\mathbf{2}})$.

(44) For all up-complete non empty posets S, T and for every element s of S and for every element t of T holds $[: ⇣s, ⇣t :] = ⇣\langle s, t \rangle$.

(45) Let S, T be antisymmetric up-complete non empty reflexive relational structures and x be an element of $[: S, T :]$. Then $\pi_1(⇣x) \subseteq ⇣(x_{\mathbf{1}})$ and $\pi_2(⇣x) \subseteq ⇣(x_{\mathbf{2}})$.

(46) Let S be an up-complete non empty poset, T be an up-complete lower-bounded non empty poset, and x be an element of $[: S, T :]$. Then $\pi_1(⇣x) = ⇣(x_{\mathbf{1}})$.

(47) Let S be an up-complete lower-bounded non empty poset, T be an up-complete non empty poset, and x be an element of $[: S, T :]$. Then $\pi_2(⇣x) = ⇣(x_{\mathbf{2}})$.

(48) For all up-complete non empty posets S, T and for every element s of S and for every element t of T holds $[: ⇡s, ⇡t :] = ⇡\langle s, t \rangle$.

(49) Let S, T be antisymmetric up-complete non empty reflexive relational structures and x be an element of $[: S, T :]$. Then $\pi_1(⇡x) \subseteq ⇡(x_{\mathbf{1}})$ and $\pi_2(⇡x) \subseteq ⇡(x_{\mathbf{2}})$.

(50) For all up-complete non empty posets S, T and for every element s of S and for every element t of T holds $[: \mathrm{compactbelow}(s), \mathrm{compactbelow}(t) :] = \mathrm{compactbelow}(\langle s, t \rangle)$.

(51) Let S, T be antisymmetric up-complete non empty reflexive relational structures and x be an element of $[: S, T :]$. Then $\pi_1(\mathrm{compactbelow}(x)) \subseteq \mathrm{compactbelow}(x_{\mathbf{1}})$ and $\pi_2(\mathrm{compactbelow}(x)) \subseteq \mathrm{compactbelow}(x_{\mathbf{2}})$.

(52) Let S be an up-complete non empty poset, T be an up-complete lower-bounded non empty poset, and x be an element of $[: S, T :]$. Then $\pi_1(\mathrm{compactbelow}(x)) = \mathrm{compactbelow}(x_{\mathbf{1}})$.

(53) Let S be an up-complete lower-bounded non empty poset, T be an up-complete non empty poset, and x be an element of $[: S, T :]$. Then $\pi_2(\mathrm{compactbelow}(x)) = \mathrm{compactbelow}(x_{\mathbf{2}})$.

Let S be a non empty reflexive relational structure. Note that every subset of S which is empty is also open.

One can prove the following propositions:

(54) Let S, T be antisymmetric up-complete non empty reflexive relational structures and X be a subset of $[: S, T :]$. If X is open, then $\pi_1(X)$ is open and $\pi_2(X)$ is open.

(55) Let S, T be up-complete non empty posets, X be a subset of S, and Y be a subset of T. If X is open and Y is open, then $[: X, Y :]$ is open.

(56) Let S, T be antisymmetric up-complete non empty reflexive relational structures and X be a subset of $[: S, T :]$. If X is inaccessible, then $\pi_1(X)$ is inaccessible and $\pi_2(X)$ is inaccessible.

(57) Let S, T be antisymmetric up-complete non empty reflexive relational structures, X be an upper subset of S, and Y be an upper subset of T. If X is inaccessible and Y is inaccessible, then $[: X, Y :]$ is inaccessible.

(58) Let S, T be antisymmetric up-complete non empty reflexive relational structures, X be a subset of S, and Y be a subset of T such that $[: X, Y :]$ is directly closed. Then

(i) if $Y \neq \emptyset$, then X is directly closed, and

(ii) if $X \neq \emptyset$, then Y is directly closed.

(59) Let S, T be antisymmetric up-complete non empty reflexive relational structures, X be a subset of S, and Y be a subset of T. Suppose X is directly closed and Y is directly closed. Then $[: X, Y :]$ is directly closed.

(60) Let S, T be antisymmetric up-complete non empty reflexive relational structures and X be a subset of $[:S, T:]$. If X has the property (S), then $\pi_1(X)$ has the property (S) and $\pi_2(X)$ has the property (S).

(61) Let S, T be up-complete non empty posets, X be a subset of S, and Y be a subset of T. If X has the property (S) and Y has the property (S), then $[:X, Y:]$ has the property (S).

3. On the Products of Relational Structures

We now state the proposition

(62) Let S, T be non empty reflexive relational structures. Suppose the relational structure of $S =$ the relational structure of T and S is inf-complete. Then T is inf-complete.

Let S be an inf-complete non empty reflexive relational structure. Observe that the relational structure of S is inf-complete.

Let S, T be inf-complete non empty reflexive relational structures. Observe that $[:S, T:]$ is inf-complete.

The following proposition is true

(63) Let S, T be non empty reflexive relational structures. If $[:S, T:]$ is inf-complete, then S is inf-complete and T is inf-complete.

Let S, T be complemented bounded antisymmetric non empty relational structures with g.l.b.'s and l.u.b.'s. One can verify that $[:S, T:]$ is complemented.

We now state the proposition

(64) Let S, T be bounded antisymmetric relational structures with g.l.b.'s and l.u.b.'s. If $[:S, T:]$ is complemented, then S is complemented and T is complemented.

Let S, T be distributive antisymmetric non empty relational structures with g.l.b.'s and l.u.b.'s. Note that $[:S, T:]$ is distributive.

The following propositions are true:

(65) Let S be an antisymmetric relational structure with g.l.b.'s and l.u.b.'s and T be a reflexive antisymmetric relational structure with g.l.b.'s and l.u.b.'s. If $[:S, T:]$ is distributive, then S is distributive.

(66) Let S be a reflexive antisymmetric relational structure with g.l.b.'s and l.u.b.'s and T be an antisymmetric relational structure with g.l.b.'s and l.u.b.'s. If $[:S, T:]$ is distributive, then T is distributive.

Let S, T be meet-continuous semilattices. Note that $[:S, T:]$ satisfies MC.

Next we state the proposition

(67) For all semilattices S, T such that $[:S, T:]$ is meet-continuous holds S is meet-continuous and T is meet-continuous.

Let S, T be up-complete inf-complete non empty posets satisfying axiom of approximation. Note that $[:S, T:]$ satisfies axiom of approximation.

Let S, T be continuous inf-complete non empty posets. Observe that $[:S, T:]$ is continuous.

One can prove the following proposition

(68) Let S, T be up-complete lower-bounded non empty posets. If $[:S, T:]$ is continuous, then S is continuous and T is continuous.

Let S, T be up-complete lower-bounded sup-semilattices satisfying axiom K. One can verify that $[:S, T:]$ satisfies axiom K.

Let S, T be complete algebraic lower-bounded sup-semilattices. Note that $[:S, T:]$ is algebraic.

The following proposition is true

(69) For all lower-bounded non empty posets S, T such that $[:S, T:]$ is algebraic holds S is algebraic and T is algebraic.

Let S, T be arithmetic lower-bounded lattices. One can check that $[:S, T:]$ is arithmetic.
One can prove the following proposition

(70) For all lower-bounded lattices S, T such that $[:S, T:]$ is arithmetic holds S is arithmetic and T is arithmetic.

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