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The "Way-Below" Relation[1]

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Summary. In the paper the "way-below" relation, in symbols $x \ll y$, is introduced. Some authors prefer the term "relatively compact" or "way inside", since in the poset of open sets of a topology it is natural to read $U \ll V$ as "U is relatively compact in V". A compact element of a poset (or an element isolated from below) is defined to be way below itself. So, the compactness in the poset of open sets of a topology is equivalent to the compactness in that topology.

The article includes definitions, facts and examples 1.1–1.8 presented in [11, pp. 38–42].

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The articles [15], [19], [20], [10], [6], [7], [16], [1], [18], [17], [14], [21], [9], [8], [5], [2], [3], [12], [4], and [13] provide the notation and terminology for this paper.

1. The "Way-Below" Relation

Let L be a non empty reflexive relational structure and let x, y be elements of L. We say that x is way below y if and only if:

(Def. 1) For every non empty directed subset D of L such that $y \leq \sup D$ there exists an element d of L such that $d \in D$ and $x \leq d$.

We introduce $x \ll y$ and $y \gg x$ as synonyms of x is way below y.

Let L be a non empty reflexive relational structure and let x be an element of L. We say that x is compact if and only if:

(Def. 2) x is way below x.

We introduce x is isolated from below as a synonym of x is compact.

Next we state several propositions:

(1) Let L be a non empty reflexive antisymmetric relational structure and x, y be elements of L. If $x \ll y$, then $x \leq y$.

(2) Let L be a non empty reflexive transitive relational structure and u, x, y, z be elements of L. If $u \leq x$ and $x \ll y$ and $y \leq z$, then $u \ll z$.

(3) Let L be a non empty poset. Suppose L is inf-complete and has l.u.b.'s. Let x, y, z be elements of L. If $x \ll z$ and $y \ll z$, then $\sup \{x, y\}$ exists in L and $x \sqcup y \ll z$.

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(4) Let L be a lower-bounded antisymmetric reflexive non empty relational structure and x be an element of L. Then $\bot_L \ll x$.

(5) For every non empty poset L and for all elements x, y, z of L such that $x \ll y$ and $y \ll z$ holds $x \ll z$.

(6) Let L be a non empty reflexive antisymmetric relational structure and x, y be elements of L. If $x \ll y$ and $x \gg y$, then $x = y$.

Let L be a non empty reflexive relational structure and let x be an element of L. The functor $\Downarrow x$ yields a subset of L and is defined by:

(Def. 3) $\Downarrow x = \{y; y$ ranges over elements of L: $y \ll x\}$.

The functor $\Uparrow x$ yielding a subset of L is defined as follows:

(Def. 4) $\Uparrow x = \{y; y$ ranges over elements of L: $y \gg x\}$.

Next we state several propositions:

(7) For every non empty reflexive relational structure L and for all elements x, y of L holds $x \in \Downarrow y$ iff $x \ll y$.

(8) For every non empty reflexive relational structure L and for all elements x, y of L holds $x \in \Uparrow y$ iff $x \gg y$.

(9) For every non empty reflexive antisymmetric relational structure L and for every element x of L holds $x \geq \Downarrow x$.

(10) For every non empty reflexive antisymmetric relational structure L and for every element x of L holds $x \leq \Uparrow x$.

(11) Let L be a non empty reflexive antisymmetric relational structure and x be an element of L. Then $\Downarrow x \subseteq \downarrow x$ and $\Uparrow x \subseteq \uparrow x$.

(12) Let L be a non empty reflexive transitive relational structure and x, y be elements of L. If $x \leq y$, then $\Downarrow x \subseteq \Downarrow y$ and $\Uparrow y \subseteq \Uparrow x$.

Let L be a lower-bounded non empty reflexive antisymmetric relational structure and let x be an element of L. One can check that $\Downarrow x$ is non empty.

Let L be a non empty reflexive transitive relational structure and let x be an element of L. Note that $\Downarrow x$ is lower and $\Uparrow x$ is upper.

Let L be a sup-semilattice and let x be an element of L. Note that $\Downarrow x$ is directed.

Let L be an inf-complete non empty poset and let x be an element of L. Observe that $\Downarrow x$ is directed.

Let L be a connected non empty relational structure. Note that every subset of L is directed and filtered.

Let us note that every non empty chain which is up-complete and lower-bounded is also complete.

Let us note that there exists a non empty chain which is complete.

The following propositions are true:

(13) For every up-complete non empty chain L and for all elements x, y of L such that $x < y$ holds $x \ll y$.

(14) Let L be a non empty reflexive antisymmetric relational structure and x, y be elements of L. If x is not compact and $x \ll y$, then $x < y$.

(15) For every non empty lower-bounded reflexive antisymmetric relational structure L holds $\bot_L$ is compact.

(16) For every up-complete non empty poset L and for every non empty finite directed subset D of L holds $\sup D \in D$.

(17) For every up-complete non empty poset L such that L is finite holds every element of L is isolated from below.

2. The Way-Below Relation in Other Terms

The scheme *SSubsetEx* deals with a non empty relational structure $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

There exists a subset X of $\mathcal{A}$ such that for every element x of $\mathcal{A}$ holds $x \in X$ iff $\mathcal{P}[x]$

for all values of the parameters.

Next we state several propositions:

(18) Let L be a complete lattice and x, y be elements of L. Suppose $x \ll y$. Let X be a subset of L. If $y \leq \sup X$, then there exists a finite subset A of L such that $A \subseteq X$ and $x \leq \sup A$.

(19) Let L be a complete lattice and x, y be elements of L. Suppose that for every subset X of L such that $y \leq \sup X$ there exists a finite subset A of L such that $A \subseteq X$ and $x \leq \sup A$. Then $x \ll y$.

(20) Let L be a non empty reflexive transitive relational structure and x, y be elements of L. If $x \ll y$, then for every ideal I of L such that $y \leq \sup I$ holds $x \in I$.

(21) Let L be an up-complete non empty poset and x, y be elements of L. If for every ideal I of L such that $y \leq \sup I$ holds $x \in I$, then $x \ll y$.

(22) Let L be a lower-bounded lattice. Suppose L is meet-continuous. Let x, y be elements of L. Then $x \ll y$ if and only if for every ideal I of L such that $y = \sup I$ holds $x \in I$.

(23) Let L be a complete lattice. Then every element of L is compact if and only if for every non empty subset X of L there exists an element x of L such that $x \in X$ and for every element y of L such that $y \in X$ holds $x \not< y$.

3. Continuous Lattices

Let L be a non empty reflexive relational structure. We say that L satisfies axiom of approximation if and only if:

(Def. 5) For every element x of L holds $x = \sup \twoheaddownarrow x$.

Let us mention that every non empty reflexive relational structure which is trivial satisfies also axiom of approximation.

Let L be a non empty reflexive relational structure. We say that L is continuous if and only if:

(Def. 6) For every element x of L holds $\twoheaddownarrow x$ is non empty and directed and L is up-complete and satisfies axiom of approximation.

Let us mention that every non empty reflexive relational structure which is continuous is also up-complete and satisfies axiom of approximation and every lower-bounded sup-semilattice which is up-complete satisfies axiom of approximation is also continuous.

Let us note that there exists a lattice which is continuous, complete, and strict.

Let L be a continuous non empty reflexive relational structure and let x be an element of L. Note that $\twoheaddownarrow x$ is non empty and directed.

We now state two propositions:

(24) Let L be an up-complete semilattice. Suppose that for every element x of L holds $\twoheaddownarrow x$ is non empty and directed. Then L satisfies axiom of approximation if and only if for all elements x, y of L such that $x \not\leq y$ there exists an element u of L such that $u \ll x$ and $u \not\leq y$.

(25) For every continuous lattice L and for all elements x, y of L holds $x \leq y$ iff $\twoheaddownarrow x \subseteq \twoheaddownarrow y$.

Let us note that every non empty chain which is complete satisfies also axiom of approximation.

The following proposition is true

(26) For every complete lattice L such that every element of L is compact holds L satisfies axiom of approximation.

4. The Way-Below Relation in Direct Powers

Let f be a binary relation. We say that f is nonempty if and only if:

(Def. 7) For every 1-sorted structure S such that $S \in \operatorname{rng} f$ holds S is non empty.

We say that f is reflexive-yielding if and only if:

(Def. 8) For every relational structure S such that $S \in \operatorname{rng} f$ holds S is reflexive.

Let I be a set. Note that there exists a many sorted set indexed by I which is relational structure yielding, nonempty, and reflexive-yielding.

Let I be a set and let J be a relational structure yielding nonempty many sorted set indexed by I. Observe that $\prod J$ is non empty.

Let I be a non empty set, let J be a relational structure yielding nonempty many sorted set indexed by I, and let i be an element of I. Then $J(i)$ is a non empty relational structure.

Let I be a set and let J be a relational structure yielding nonempty many sorted set indexed by I. Note that every element of $\prod J$ is function-like and relation-like.

Let I be a non empty set, let J be a relational structure yielding nonempty many sorted set indexed by I, let x be an element of $\prod J$, and let i be an element of I. Then $x(i)$ is an element of $J(i)$.

Let I be a non empty set, let J be a relational structure yielding nonempty many sorted set indexed by I, let i be an element of I, and let X be a subset of $\prod J$. Then $\pi_i X$ is a subset of $J(i)$.

Next we state two propositions:

(27) Let I be a non empty set, J be a relational structure yielding nonempty many sorted set indexed by I, and x be a function. Then x is an element of $\prod J$ if and only if $\operatorname{dom} x = I$ and for every element i of I holds $x(i)$ is an element of $J(i)$.

(28) Let I be a non empty set, J be a relational structure yielding nonempty many sorted set indexed by I, and x, y be elements of $\prod J$. Then $x \leq y$ if and only if for every element i of I holds $x(i) \leq y(i)$.

Let I be a non empty set and let J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Observe that $\prod J$ is reflexive. Let i be an element of I. Then $J(i)$ is a non empty reflexive relational structure.

Let I be a non empty set, let J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I, let x be an element of $\prod J$, and let i be an element of I. Then $x(i)$ is an element of $J(i)$.

The following propositions are true:

(29) Let I be a non empty set and J be a relational structure yielding nonempty many sorted set indexed by I. If for every element i of I holds $J(i)$ is transitive, then $\prod J$ is transitive.

(30) Let I be a non empty set and J be a relational structure yielding nonempty many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is antisymmetric. Then $\prod J$ is antisymmetric.

(31) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a complete lattice. Then $\prod J$ is a complete lattice.

(32) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a complete lattice. Let X be a subset of $\prod J$ and i be an element of I. Then $(\sup X)(i) = \sup \pi_i X$.

(33) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is a complete lattice. Let x, y be elements of $\prod J$. Then $x \ll y$ if and only if the following conditions are satisfied:

(i) for every element i of I holds $x(i) \ll y(i)$, and

(ii) there exists a finite subset K of I such that for every element i of I such that $i \notin K$ holds $x(i) = \perp_{J(i)}$.

5. THE WAY-BELOW RELATION IN TOPOLOGICAL SPACES

We now state four propositions:

(34) Let T be a non empty topological space and x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. Suppose x is way below y. Let F be a family of subsets of T. If F is open and $y \subseteq \bigcup F$, then there exists a finite subset G of F such that $x \subseteq \bigcup G$.

(35) Let T be a non empty topological space and x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. Suppose that for every family F of subsets of T such that F is open and $y \subseteq \bigcup F$ there exists a finite subset G of F such that $x \subseteq \bigcup G$. Then x is way below y.

(36) Let T be a non empty topological space, x be an element of $\langle$the topology of T, $\subseteq\rangle$, and X be a subset of T. If $x = X$, then x is compact iff X is compact.

(37) Let T be a non empty topological space and x be an element of $\langle$the topology of T, $\subseteq\rangle$. Suppose $x =$ the carrier of T. Then x is compact if and only if T is compact.

Let T be a non empty topological space. We say that T is locally-compact if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let x be a point of T and X be a subset of T. Suppose $x \in X$ and X is open. Then there exists a subset Y of T such that $x \in \operatorname{Int} Y$ and $Y \subseteq X$ and Y is compact.

Let us observe that every non empty topological space which is compact and T_2 is also T_3, T_4, and locally-compact.

One can prove the following proposition

(38) For every set x holds $\{\{x\}\}_{\mathrm{top}}$ is T_2.

Let us observe that there exists a non empty topological space which is compact and T_2.

Next we state several propositions:

(39) Let T be a non empty topological space and x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. If there exists a subset Z of T such that $x \subseteq Z$ and $Z \subseteq y$ and Z is compact, then $x \ll y$.

(40) Let T be a non empty topological space. Suppose T is locally-compact. Let x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. If $x \ll y$, then there exists a subset Z of T such that $x \subseteq Z$ and $Z \subseteq y$ and Z is compact.

(41) Let T be a non empty topological space. Suppose T is locally-compact and a T_2 space. Let x, y be elements of $\langle$the topology of T, $\subseteq\rangle$. If $x \ll y$, then there exists a subset Z of T such that $Z = x$ and $\overline{Z} \subseteq y$ and $\overline{Z}$ is compact.

(42) Let X be a non empty topological space. Suppose X is a T_3 space and $\langle$the topology of X, $\subseteq\rangle$ is continuous. Then X is locally-compact.

(43) For every non empty topological space T such that T is locally-compact holds $\langle$the topology of T, $\subseteq\rangle$ is continuous.

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