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# Meet – Continuous Lattices[1]

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**Summary.** The aim of this work is the formalization of Chapter 0 Section 4 of [11]. In this paper the definition of meet-continuous lattices is introduced. Theorem 4.2 and Remark 4.3 are proved.

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The articles [18], [8], [21], [22], [19], [5], [14], [10], [7], [6], [17], [4], [20], [12], [1], [2], [3], [13], [23], [9], [15], and [16] provide the notation and terminology for this paper.

## 1. Preliminaries

Let $X$, $Y$ be non empty sets, let $f$ be a function from $X$ into $Y$, and let $Z$ be a non empty subset of $X$. One can verify that $f^\circ Z$ is non empty.

Let us note that every non empty relational structure which is reflexive and connected has also g.l.b.'s and l.u.b.'s.

Let $C$ be a chain. Note that $\Omega_C$ is directed.

Next we state a number of propositions:

(1) Let $L$ be an up-complete semilattice, $D$ be a non empty directed subset of $L$, and $x$ be an element of $L$. Then $\sup \{x\} \sqcap D$ exists in $L$.

(2) Let $L$ be an up-complete sup-semilattice, $D$ be a non empty directed subset of $L$, and $x$ be an element of $L$. Then $\sup \{x\} \sqcup D$ exists in $L$.

(3) For every up-complete sup-semilattice $L$ and for all non empty directed subsets $A$, $B$ of $L$ holds $A \leq \sup(A \sqcup B)$.

(4) For every up-complete sup-semilattice $L$ and for all non empty directed subsets $A$, $B$ of $L$ holds $\sup(A \sqcup B) = \sup A \sqcup \sup B$.

(5) Let $L$ be an up-complete semilattice and $D$ be a non empty directed subset of $[:L, L:]$. Then $\{\sup(\{x\} \sqcap \pi_2(D)); x$ ranges over elements of $L$: $x \in \pi_1(D)\} = \{\sup X; X$ ranges over non empty directed subsets of $L$: $\bigvee_{x:\text{element of } L} (X = \{x\} \sqcap \pi_2(D) \wedge x \in \pi_1(D))\}$.

(6) Let $L$ be a semilattice and $D$ be a non empty directed subset of $[:L, L:]$. Then $\bigcup\{X; X$ ranges over non empty directed subsets of $L$: $\bigvee_{x:\text{element of } L} (X = \{x\} \sqcap \pi_2(D) \wedge x \in \pi_1(D))\} = \pi_1(D) \sqcap \pi_2(D)$.

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(7) Let $L$ be an up-complete semilattice and $D$ be a non empty directed subset of $[:L, L:]$. Then $\sup \bigcup \{X; X$ ranges over non empty directed subsets of $L$: $\bigvee_{x:\text{element of } L} (X = \{x\} \sqcap \pi_2(D) \wedge x \in \pi_1(D))\}$ exists in $L$.

(8) Let $L$ be an up-complete semilattice and $D$ be a non empty directed subset of $[:L, L:]$. Then $\sup \{\sup X; X$ ranges over non empty directed subsets of $L$: $\bigvee_{x:\text{element of } L} (X = \{x\} \sqcap \pi_2(D) \wedge x \in \pi_1(D))\}$ exists in $L$.

(9) Let $L$ be an up-complete semilattice and $D$ be a non empty directed subset of $[:L, L:]$. Then $\bigsqcup_L \{\sup X; X$ ranges over non empty directed subsets of $L$: $\bigvee_{x:\text{element of } L} (X = \{x\} \sqcap \pi_2(D) \wedge x \in \pi_1(D))\} \leq \bigsqcup_L \bigcup \{X; X$ ranges over non empty directed subsets of $L$: $\bigvee_{x:\text{element of } L} (X = \{x\} \sqcap \pi_2(D) \wedge x \in \pi_1(D))\}$.

(10) Let $L$ be an up-complete semilattice and $D$ be a non empty directed subset of $[:L, L:]$. Then $\bigsqcup_L \{\sup X; X$ ranges over non empty directed subsets of $L$: $\bigvee_{x:\text{element of } L} (X = \{x\} \sqcap \pi_2(D) \wedge x \in \pi_1(D))\} = \bigsqcup_L \bigcup \{X; X$ ranges over non empty directed subsets of $L$: $\bigvee_{x:\text{element of } L} (X = \{x\} \sqcap \pi_2(D) \wedge x \in \pi_1(D))\}$.

Let $S$, $T$ be up-complete non empty reflexive relational structures. Observe that $[:S, T:]$ is up-complete.

The following four propositions are true:

(11) Let $S$, $T$ be non empty reflexive antisymmetric relational structures. If $[:S, T:]$ is up-complete, then $S$ is up-complete and $T$ is up-complete.

(12) Let $L$ be an up-complete antisymmetric non empty reflexive relational structure and $D$ be a non empty directed subset of $[:L, L:]$. Then $\sup D = \langle \sup \pi_1(D), \sup \pi_2(D) \rangle$.

(13) Let $S_1$, $S_2$ be relational structures, $D$ be a subset of $S_1$, and $f$ be a map from $S_1$ into $S_2$. If $f$ is monotone, then $f^\circ {\downarrow} D \subseteq {\downarrow}(f^\circ D)$.

(14) Let $S_1$, $S_2$ be relational structures, $D$ be a subset of $S_1$, and $f$ be a map from $S_1$ into $S_2$. If $f$ is monotone, then $f^\circ {\uparrow} D \subseteq {\uparrow}(f^\circ D)$.

One can check that every non empty reflexive relational structure which is trivial is also distributive and complemented.

One can check that there exists a lattice which is strict, non empty, and trivial.

Next we state three propositions:

(15) Let $H$ be a distributive complete lattice, $a$ be an element of $H$, and $X$ be a finite subset of $H$. Then $\sup(\{a\} \sqcap X) = a \sqcap \sup X$.

(16) Let $H$ be a distributive complete lattice, $a$ be an element of $H$, and $X$ be a finite subset of $H$. Then $\inf(\{a\} \sqcup X) = a \sqcup \inf X$.

(17) Let $H$ be a complete distributive lattice, $a$ be an element of $H$, and $X$ be a finite subset of $H$. Then $a \sqcap \square$ preserves sup of $X$.

## 2. THE PROPERTIES OF NETS

The scheme *ExNet* deals with a non empty relational structure $\mathcal{A}$, a prenet $\mathcal{B}$ over $\mathcal{A}$, and a unary functor $\mathcal{F}$ yielding an element of $\mathcal{A}$, and states that:

There exists a prenet $M$ over $\mathcal{A}$ such that

(i) the relational structure of $M =$ the relational structure of $\mathcal{B}$, and

(ii) for every element $i$ of $M$ holds (the mapping of $M)(i) = \mathcal{F}(($the mapping of $\mathcal{B})(i))$

for all values of the parameters.

The following propositions are true:

(18) Let $L$ be a non empty relational structure and $N$ be a prenet over $L$. If $N$ is eventually-directed, then $\operatorname{rng} \operatorname{netmap}(N, L)$ is directed.

(19) Let $L$ be a non empty reflexive relational structure, $D$ be a non empty directed subset of $L$, and $n$ be a function from $D$ into the carrier of $L$. Then $\langle D, (\text{the internal relation of } L) \mid^2 D, n\rangle$ is a prenet over $L$.

(20) Let $L$ be a non empty reflexive relational structure, $D$ be a non empty directed subset of $L$, $n$ be a function from $D$ into the carrier of $L$, and $N$ be a prenet over $L$. Suppose $n = \mathrm{id}_D$ and $N = \langle D, (\text{the internal relation of } L) \mid^2 D, n\rangle$. Then $N$ is eventually-directed.

Let $L$ be a non empty relational structure and let $N$ be a net structure over $L$. The functor $\sup N$ yielding an element of $L$ is defined as follows:

(Def. 1) $\sup N = \operatorname{Sup}(\text{the mapping of } N)$.

Let $L$ be a non empty relational structure, let $J$ be a set, and let $f$ be a function from $J$ into the carrier of $L$. The functor $\operatorname{FinSups}(f)$ yields a prenet over $L$ and is defined by the condition (Def. 2).

(Def. 2) There exists a function $g$ from $\operatorname{Fin} J$ into the carrier of $L$ such that for every element $x$ of $\operatorname{Fin} J$ holds

$g(x) = \sup(f^\circ x)$ and $\operatorname{FinSups}(f) = \langle \operatorname{Fin} J, \subseteq_{\operatorname{Fin} J}, g\rangle$.

One can prove the following proposition

(21) Let $L$ be a non empty relational structure, $J$, $x$ be sets, and $f$ be a function from $J$ into the carrier of $L$. Then $x$ is an element of $\operatorname{FinSups}(f)$ if and only if $x$ is an element of $\operatorname{Fin} J$.

Let $L$ be a complete antisymmetric non empty reflexive relational structure, let $J$ be a set, and let $f$ be a function from $J$ into the carrier of $L$. Observe that $\operatorname{FinSups}(f)$ is monotone.

Let $L$ be a non empty relational structure, let $x$ be an element of $L$, and let $N$ be a non empty net structure over $L$. The functor $x \sqcap N$ yields a strict net structure over $L$ and is defined by the conditions (Def. 3).

(Def. 3)(i) The relational structure of $x \sqcap N =$ the relational structure of $N$, and

(ii) for every element $i$ of $x \sqcap N$ there exists an element $y$ of $L$ such that $y = (\text{the mapping of } N)(i)$ and $(\text{the mapping of } x \sqcap N)(i) = x \sqcap y$.

Next we state the proposition

(22) Let $L$ be a non empty relational structure, $N$ be a non empty net structure over $L$, $x$ be an element of $L$, and $y$ be a set. Then $y$ is an element of $N$ if and only if $y$ is an element of $x \sqcap N$.

Let $L$ be a non empty relational structure, let $x$ be an element of $L$, and let $N$ be a non empty net structure over $L$. Note that $x \sqcap N$ is non empty.

Let $L$ be a non empty relational structure, let $x$ be an element of $L$, and let $N$ be a prenet over $L$. One can check that $x \sqcap N$ is directed.

We now state several propositions:

(23) Let $L$ be a non empty relational structure, $x$ be an element of $L$, and $F$ be a non empty net structure over $L$. Then $\operatorname{rng}(\text{the mapping of } x \sqcap F) = \{x\} \sqcap \operatorname{rng}(\text{the mapping of } F)$.

(24) Let $L$ be a non empty relational structure, $J$ be a set, and $f$ be a function from $J$ into the carrier of $L$. If for every set $x$ holds $\sup f^\circ x$ exists in $L$, then $\operatorname{rng} \operatorname{netmap}(\operatorname{FinSups}(f), L) \subseteq \operatorname{finsups}(\operatorname{rng} f)$.

(25) Let $L$ be a non empty reflexive antisymmetric relational structure, $J$ be a set, and $f$ be a function from $J$ into the carrier of $L$. Then $\operatorname{rng} f \subseteq \operatorname{rng} \operatorname{netmap}(\operatorname{FinSups}(f), L)$.

(26) Let $L$ be a non empty reflexive antisymmetric relational structure, $J$ be a set, and $f$ be a function from $J$ into the carrier of $L$. Suppose sup $\operatorname{rng} f$ exists in $L$ and sup $\operatorname{rng}\operatorname{netmap}(\operatorname{FinSups}(f), L)$ exists in $L$ and for every element $x$ of $\operatorname{Fin} J$ holds sup $f^{\circ}x$ exists in $L$. Then $\operatorname{Sup}(f) = \sup \operatorname{FinSups}(f)$.

(27) Let $L$ be an antisymmetric transitive relational structure with g.l.b.'s, $N$ be a prenet over $L$, and $x$ be an element of $L$. If $N$ is eventually-directed, then $x \sqcap N$ is eventually-directed.

(28) Let $L$ be an up-complete semilattice. Suppose that for every element $x$ of $L$ and for every non empty directed subset $E$ of $L$ such that $x \leq \sup E$ holds $x \leq \sup(\{x\} \sqcap E)$. Let $D$ be a non empty directed subset of $L$ and $x$ be an element of $L$. Then $x \sqcap \sup D = \sup(\{x\} \sqcap D)$.

(29) Let $L$ be a poset with l.u.b.'s. Suppose that for every directed subset $X$ of $L$ and for every element $x$ of $L$ holds $x \sqcap \sup X = \sup(\{x\} \sqcap X)$. Let $X$ be a subset of $L$ and $x$ be an element of $L$. If sup $X$ exists in $L$, then $x \sqcap \sup X = \sup(\{x\} \sqcap \operatorname{finsups}(X))$.

(30) Let $L$ be an up-complete lattice. Suppose that for every subset $X$ of $L$ and for every element $x$ of $L$ holds $x \sqcap \sup X = \sup(\{x\} \sqcap \operatorname{finsups}(X))$. Let $X$ be a non empty directed subset of $L$ and $x$ be an element of $L$. Then $x \sqcap \sup X = \sup(\{x\} \sqcap X)$.

## 3. On the inf and sup operation

Let $L$ be a non empty relational structure. The functor $\sqcap_L$ yields a map from $[:L, L:]$ into $L$ and is defined by:

(Def. 4) For all elements $x$, $y$ of $L$ holds $\sqcap_L(\langle x, y \rangle) = x \sqcap y$.

One can prove the following proposition

(31) For every non empty relational structure $L$ and for every element $x$ of $[:L, L:]$ holds $\sqcap_L(x) = x_{\mathbf{1}} \sqcap x_{\mathbf{2}}$.

Let $L$ be a transitive antisymmetric relational structure with g.l.b.'s. Observe that $\sqcap_L$ is monotone.

Next we state two propositions:

(32) For every non empty relational structure $S$ and for all subsets $D_1$, $D_2$ of $S$ holds $(\sqcap_S)^{\circ}[:D_1, D_2:] = D_1 \sqcap D_2$.

(33) For every up-complete semilattice $L$ and for every non empty directed subset $D$ of $[:L, L:]$ holds $\sup((\sqcap_L)^{\circ}D) = \sup(\pi_1(D) \sqcap \pi_2(D))$.

Let $L$ be a non empty relational structure. The functor $\sqcup_L$ yielding a map from $[:L, L:]$ into $L$ is defined as follows:

(Def. 5) For all elements $x$, $y$ of $L$ holds $\sqcup_L(\langle x, y \rangle) = x \sqcup y$.

We now state the proposition

(34) For every non empty relational structure $L$ and for every element $x$ of $[:L, L:]$ holds $\sqcup_L(x) = x_{\mathbf{1}} \sqcup x_{\mathbf{2}}$.

Let $L$ be a transitive antisymmetric relational structure with l.u.b.'s. Note that $\sqcup_L$ is monotone.

Next we state two propositions:

(35) For every non empty relational structure $S$ and for all subsets $D_1$, $D_2$ of $S$ holds $(\sqcup_S)^{\circ}[:D_1, D_2:] = D_1 \sqcup D_2$.

(36) For every complete non empty poset $L$ and for every non empty filtered subset $D$ of $[:L, L:]$ holds $\inf((\sqcup_L)^{\circ}D) = \inf(\pi_1(D) \sqcup \pi_2(D))$.

## 4. Meet-continuous lattices

Let $R$ be a non empty reflexive relational structure. We say that $R$ satisfies MC if and only if:

(Def. 6) For every element $x$ of $R$ and for every non empty directed subset $D$ of $R$ holds $x \sqcap \sup D = \sup(\{x\} \sqcap D)$.

Let $R$ be a non empty reflexive relational structure. We say that $R$ is meet-continuous if and only if:

(Def. 7) $R$ is up-complete and satisfies MC.

Let us note that every non empty reflexive relational structure which is trivial satisfies also MC.

Let us observe that every non empty reflexive relational structure which is meet-continuous is also up-complete and satisfies MC and every non empty reflexive relational structure which is up-complete satisfies MC is also meet-continuous.

Let us observe that there exists a lattice which is strict, non empty, and trivial.

One can prove the following two propositions:

(37) Let $S$ be a non empty reflexive relational structure such that for every subset $X$ of $S$ and for every element $x$ of $S$ holds $x \sqcap \sup X = \bigsqcup_S \{x \sqcap y; y$ ranges over elements of $S$: $y \in X\}$. Then $S$ satisfies MC.

(38) Let $L$ be an up-complete semilattice. If $\mathrm{SupMap}(L)$ is meet-preserving, then for all ideals $I_1$, $I_2$ of $L$ holds $\sup I_1 \sqcap \sup I_2 = \sup(I_1 \sqcap I_2)$.

Let $L$ be an up-complete sup-semilattice. One can check that $\mathrm{SupMap}(L)$ is join-preserving.

One can prove the following propositions:

(39) Let $L$ be an up-complete semilattice. If for all ideals $I_1$, $I_2$ of $L$ holds $\sup I_1 \sqcap \sup I_2 = \sup(I_1 \sqcap I_2)$, then $\mathrm{SupMap}(L)$ is meet-preserving.

(40) Let $L$ be an up-complete semilattice. Suppose that for all ideals $I_1$, $I_2$ of $L$ holds $\sup I_1 \sqcap \sup I_2 = \sup(I_1 \sqcap I_2)$. Let $D_1$, $D_2$ be directed non empty subsets of $L$. Then $\sup D_1 \sqcap \sup D_2 = \sup(D_1 \sqcap D_2)$.

(41) Let $L$ be a non empty reflexive relational structure. Suppose $L$ satisfies MC. Let $x$ be an element of $L$ and $N$ be a non empty prenet over $L$. If $N$ is eventually-directed, then $x \sqcap \sup N = \sup(\{x\} \sqcap \mathrm{rng}\,\mathrm{netmap}(N, L))$.

(42) Let $L$ be a non empty reflexive relational structure such that for every element $x$ of $L$ and for every prenet $N$ over $L$ such that $N$ is eventually-directed holds $x \sqcap \sup N = \sup(\{x\} \sqcap \mathrm{rng}\,\mathrm{netmap}(N, L))$. Then $L$ satisfies MC.

(43) Let $L$ be an up-complete antisymmetric non empty reflexive relational structure. Suppose $\sqcap_L$ is directed-sups-preserving. Let $D_1$, $D_2$ be non empty directed subsets of $L$. Then $\sup D_1 \sqcap \sup D_2 = \sup(D_1 \sqcap D_2)$.

(44) Let $L$ be a non empty reflexive antisymmetric relational structure such that for all non empty directed subsets $D_1$, $D_2$ of $L$ holds $\sup D_1 \sqcap \sup D_2 = \sup(D_1 \sqcap D_2)$. Then $L$ satisfies MC.

(45) Let $L$ be an antisymmetric non empty reflexive relational structure with g.l.b.'s and satisfying MC, $x$ be an element of $L$, and $D$ be a non empty directed subset of $L$. If $x \leq \sup D$, then $x = \sup(\{x\} \sqcap D)$.

(46) Let $L$ be an up-complete semilattice. Suppose that for every element $x$ of $L$ and for every non empty directed subset $E$ of $L$ such that $x \leq \sup E$ holds $x \leq \sup(\{x\} \sqcap E)$. Then $\sqcap_L$ is directed-sups-preserving.

(47) Let $L$ be a complete antisymmetric non empty reflexive relational structure. Suppose that for every element $x$ of $L$ and for every prenet $N$ over $L$ such that $N$ is eventually-directed holds $x \sqcap \sup N = \sup(\{x\} \sqcap \operatorname{rng}\operatorname{netmap}(N, L))$. Let $x$ be an element of $L$, $J$ be a set, and $f$ be a function from $J$ into the carrier of $L$. Then $x \sqcap \operatorname{Sup}(f) = \sup(x \sqcap \operatorname{FinSups}(f))$.

(48) Let $L$ be a complete semilattice. Suppose that for every element $x$ of $L$ and for every set $J$ and for every function $f$ from $J$ into the carrier of $L$ holds $x \sqcap \operatorname{Sup}(f) = \sup(x \sqcap \operatorname{FinSups}(f))$. Let $x$ be an element of $L$ and $N$ be a prenet over $L$. If $N$ is eventually-directed, then $x \sqcap \sup N = \sup(\{x\} \sqcap \operatorname{rng}\operatorname{netmap}(N, L))$.

(49) For every up-complete lattice $L$ holds $L$ is meet-continuous iff $\operatorname{SupMap}(L)$ is meet-preserving and join-preserving.

Let $L$ be a meet-continuous lattice. One can verify that $\operatorname{SupMap}(L)$ is meet-preserving and join-preserving.

We now state four propositions:

(50) Let $L$ be an up-complete lattice. Then $L$ is meet-continuous if and only if for all ideals $I_1$, $I_2$ of $L$ holds $\sup I_1 \sqcap \sup I_2 = \sup(I_1 \sqcap I_2)$.

(51) Let $L$ be an up-complete lattice. Then $L$ is meet-continuous if and only if for all non empty directed subsets $D_1$, $D_2$ of $L$ holds $\sup D_1 \sqcap \sup D_2 = \sup(D_1 \sqcap D_2)$.

(52) Let $L$ be an up-complete lattice. Then $L$ is meet-continuous if and only if for every element $x$ of $L$ and for every non empty directed subset $D$ of $L$ such that $x \leq \sup D$ holds $x = \sup(\{x\} \sqcap D)$.

(53) For every up-complete semilattice $L$ holds $L$ is meet-continuous iff $\sqcap_L$ is directed-sups-preserving.

Let $L$ be a meet-continuous semilattice. Note that $\sqcap_L$ is directed-sups-preserving.

One can prove the following propositions:

(54) Let $L$ be an up-complete semilattice. Then $L$ is meet-continuous if and only if for every element $x$ of $L$ and for every non empty prenet $N$ over $L$ such that $N$ is eventually-directed holds $x \sqcap \sup N = \sup(\{x\} \sqcap \operatorname{rng}\operatorname{netmap}(N, L))$.

(55) Let $L$ be a complete semilattice. Then $L$ is meet-continuous if and only if for every element $x$ of $L$ and for every set $J$ and for every function $f$ from $J$ into the carrier of $L$ holds $x \sqcap \operatorname{Sup}(f) = \sup(x \sqcap \operatorname{FinSups}(f))$.

Let $L$ be a meet-continuous semilattice and let $x$ be an element of $L$. One can verify that $x \sqcap \square$ is directed-sups-preserving.

The following proposition is true

(56) For every complete non empty poset $H$ holds $H$ is Heyting iff $H$ is meet-continuous and distributive.

Let us observe that every non empty poset which is complete and Heyting is also meet-continuous and distributive and every non empty poset which is complete, meet-continuous, and distributive is also Heyting.

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