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Galois Connections [1]

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Summary. The paper is the Mizar encoding of the chapter 0 section 3 of [9] In the paper the following concept are defined: Galois connections, Heyting algebras, and Boolean algebras.

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The articles [14], [8], [16], [5], [7], [6], [13], [15], [1], [2], [10], [3], [12], [11], [4], and [17] provide the notation and terminology for this paper.

1. PRELIMINARIES

Let L_1, L_2 be non empty 1-sorted structures and let f be a map from L_1 into L_2. Let us observe that f is one-to-one if and only if:

(Def. 1) For all elements x, y of L_1 such that $f(x) = f(y)$ holds $x = y$.

Next we state the proposition

(1) Let L be a non empty 1-sorted structure and f be a map from L into L. If for every element x of L holds $f(x) = x$, then $f = \mathrm{id}_L$.

Let L_1, L_2 be non empty relational structures and let f be a map from L_1 into L_2. Let us observe that f is monotone if and only if:

(Def. 2) For all elements x, y of L_1 such that $x \leq y$ holds $f(x) \leq f(y)$.

The following four propositions are true:

(2) Let L be an antisymmetric transitive relational structure with g.l.b.'s and x, y, z be elements of L. If $x \leq y$, then $x \sqcap z \leq y \sqcap z$.

(3) Let L be an antisymmetric transitive relational structure with l.u.b.'s and x, y, z be elements of L. If $x \leq y$, then $x \sqcup z \leq y \sqcup z$.

(4) Let L be a non empty lower-bounded antisymmetric relational structure and x be an element of L. Then

(i) if L has g.l.b.'s, then $\perp_L \sqcap x = \perp_L$, and

(ii) if L is reflexive and transitive and has l.u.b.'s, then $\perp_L \sqcup x = x$.

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(5) Let L be a non empty upper-bounded antisymmetric relational structure and x be an element of L. Then

(i) if L is transitive and reflexive and has g.l.b.'s, then $\top_L \sqcap x = x$, and

(ii) if L has l.u.b.'s, then $\top_L \sqcup x = \top_L$.

Let L be a non empty relational structure. We say that L is distributive if and only if:

(Def. 3) For all elements x, y, z of L holds $x \sqcap (y \sqcup z) = (x \sqcap y) \sqcup (x \sqcap z)$.

Next we state the proposition

(6) For every lattice L holds L is distributive iff for all elements x, y, z of L holds $x \sqcup (y \sqcap z) = (x \sqcup y) \sqcap (x \sqcup z)$.

Let X be a set. Observe that $2_{\subseteq}^{X}$ is distributive.

Let S be a non empty relational structure and let X be a set. We say that min X exists in S if and only if:

(Def. 4) Inf X exists in S and $\bigsqcap_S X \in X$.

We introduce X has the minimum in S as a synonym of min X exists in S. We say that max X exists in S if and only if:

(Def. 5) Sup X exists in S and $\bigsqcup_S X \in X$.

We introduce X has the maximum in S as a synonym of max X exists in S.

Let S be a non empty relational structure, let s be an element of S, and let X be a set. We say that s is a minimum of X if and only if:

(Def. 6) Inf X exists in S and $s = \bigsqcap_S X$ and $\bigsqcap_S X \in X$.

We say that s is a maximum of X if and only if:

(Def. 7) Sup X exists in S and $s = \bigsqcup_S X$ and $\bigsqcup_S X \in X$.

Let L be a relational structure. Note that id_L is isomorphic.

Let L_1, L_2 be relational structures. We say that L_1 and L_2 are isomorphic if and only if:

(Def. 8) There exists a map from L_1 into L_2 which is isomorphic.

Let us note that the predicate L_1 and L_2 are isomorphic is reflexive.

The following two propositions are true:

(7) For all non empty relational structures L_1, L_2 such that L_1 and L_2 are isomorphic holds L_2 and L_1 are isomorphic.

(8) Let L_1, L_2, L_3 be relational structures. Suppose L_1 and L_2 are isomorphic and L_2 and L_3 are isomorphic. Then L_1 and L_3 are isomorphic.

2. Galois Connections

Let S, T be relational structures. A set is called a connection between S and T if:

(Def. 9) There exists a map g from S into T and there exists a map d from T into S such that it $= \langle g, d \rangle$.

Let S, T be relational structures, let g be a map from S into T, and let d be a map from T into S. Then $\langle g, d \rangle$ is a connection between S and T.

Let S, T be non empty relational structures and let g_1 be a connection between S and T. We say that g_1 is Galois if and only if the condition (Def. 10) is satisfied.

(Def. 10) There exists a map g from S into T and there exists a map d from T into S such that

(i) $g_1 = \langle g, d \rangle$,

(ii) g is monotone,

(iii) d is monotone, and

(iv) for every element t of T and for every element s of S holds $t \leq g(s)$ iff $d(t) \leq s$.

We now state the proposition

(9) Let S, T be non empty posets, g be a map from S into T, and d be a map from T into S. Then $\langle g, d \rangle$ is Galois if and only if the following conditions are satisfied:

(i) g is monotone,

(ii) d is monotone, and

(iii) for every element t of T and for every element s of S holds $t \leq g(s)$ iff $d(t) \leq s$.

Let S, T be non empty relational structures and let g be a map from S into T. We say that g is upper adjoint if and only if:

(Def. 11) There exists a map d from T into S such that $\langle g, d \rangle$ is Galois.

We introduce g has a lower adjoint as a synonym of g is upper adjoint.

Let S, T be non empty relational structures and let d be a map from T into S. We say that d is lower adjoint if and only if:

(Def. 12) There exists a map g from S into T such that $\langle g, d \rangle$ is Galois.

We introduce d has an upper adjoint as a synonym of d is lower adjoint.

The following four propositions are true:

(10) Let S, T be non empty posets, g be a map from S into T, and d be a map from T into S. If $\langle g, d \rangle$ is Galois, then g is upper adjoint and d is lower adjoint.

(11) Let S, T be non empty posets, g be a map from S into T, and d be a map from T into S. Then $\langle g, d \rangle$ is Galois if and only if the following conditions are satisfied:

(i) g is monotone, and

(ii) for every element t of T holds $d(t)$ is a minimum of $g^{-1}(\uparrow t)$.

(12) Let S, T be non empty posets, g be a map from S into T, and d be a map from T into S. Then $\langle g, d \rangle$ is Galois if and only if the following conditions are satisfied:

(i) d is monotone, and

(ii) for every element s of S holds $g(s)$ is a maximum of $d^{-1}(\downarrow s)$.

(13) Let S, T be non empty posets and g be a map from S into T. If g is upper adjoint, then g is infs-preserving.

Let S, T be non empty posets. Observe that every map from S into T which is upper adjoint is also infs-preserving.

Next we state the proposition

(14) Let S, T be non empty posets and d be a map from T into S. If d is lower adjoint, then d is sups-preserving.

Let S, T be non empty posets. Observe that every map from S into T which is lower adjoint is also sups-preserving.

Next we state a number of propositions:

(15) Let S, T be non empty posets and g be a map from S into T. Suppose S is complete and g is infs-preserving. Then there exists a map d from T into S such that $\langle g, d\rangle$ is Galois and for every element t of T holds $d(t)$ is a minimum of $g^{-1}(\uparrow t)$.

(16) Let S, T be non empty posets and d be a map from T into S. Suppose T is complete and d is sups-preserving. Then there exists a map g from S into T such that $\langle g, d\rangle$ is Galois and for every element s of S holds $g(s)$ is a maximum of $d^{-1}(\downarrow s)$.

(17) Let S, T be non empty posets and g be a map from S into T. Suppose S is complete. Then g is infs-preserving if and only if g is monotone and g has a lower adjoint.

(18) Let S, T be non empty posets and d be a map from T into S. Suppose T is complete. Then d is sups-preserving if and only if d is monotone and d has an upper adjoint.

(19) Let S, T be non empty posets, g be a map from S into T, and d be a map from T into S. If $\langle g, d\rangle$ is Galois, then $d \cdot g \leq \mathrm{id}_S$ and $\mathrm{id}_T \leq g \cdot d$.

(20) Let S, T be non empty posets, g be a map from S into T, and d be a map from T into S. Suppose g is monotone and d is monotone and $d \cdot g \leq \mathrm{id}_S$ and $\mathrm{id}_T \leq g \cdot d$. Then $\langle g, d\rangle$ is Galois.

(21) Let S, T be non empty posets, g be a map from S into T, and d be a map from T into S. Suppose g is monotone and d is monotone and $d \cdot g \leq \mathrm{id}_S$ and $\mathrm{id}_T \leq g \cdot d$. Then $d = d \cdot g \cdot d$ and $g = g \cdot d \cdot g$.

(22) Let S, T be non empty relational structures, g be a map from S into T, and d be a map from T into S. If $d = d \cdot g \cdot d$ and $g = g \cdot d \cdot g$, then $g \cdot d$ is idempotent and $d \cdot g$ is idempotent.

(23) Let S, T be non empty posets, g be a map from S into T, and d be a map from T into S. Suppose $\langle g, d\rangle$ is Galois and g is onto. Let t be an element of T. Then $d(t)$ is a minimum of $g^{-1}(\{t\})$.

(24) Let S, T be non empty posets, g be a map from S into T, and d be a map from T into S. If for every element t of T holds $d(t)$ is a minimum of $g^{-1}(\{t\})$, then $g \cdot d = \mathrm{id}_T$.

(25) Let L_1, L_2 be non empty 1-sorted structures, g_3 be a map from L_1 into L_2, and g_2 be a map from L_2 into L_1. If $g_2 \cdot g_3 = \mathrm{id}_{(L_1)}$, then g_3 is one-to-one and g_2 is onto.

(26) Let S, T be non empty posets, g be a map from S into T, and d be a map from T into S. If $\langle g, d\rangle$ is Galois, then g is onto iff d is one-to-one.

(27) Let S, T be non empty posets, g be a map from S into T, and d be a map from T into S. Suppose $\langle g, d\rangle$ is Galois and d is onto. Let s be an element of S. Then $g(s)$ is a maximum of $d^{-1}(\{s\})$.

(28) Let S, T be non empty posets, g be a map from S into T, and d be a map from T into S. If for every element s of S holds $g(s)$ is a maximum of $d^{-1}(\{s\})$, then $d \cdot g = \mathrm{id}_S$.

(29) Let S, T be non empty posets, g be a map from S into T, and d be a map from T into S. If $\langle g, d\rangle$ is Galois, then g is one-to-one iff d is onto.

Let L be a non empty relational structure and let p be a map from L into L. We say that p is projection if and only if:

(Def. 13) p is idempotent and monotone.

We introduce p is a projection operator as a synonym of p is projection.

Let L be a non empty relational structure. One can verify that id_L is projection.

Let L be a non empty relational structure. Note that there exists a map from L into L which is projection.

Let L be a non empty relational structure and let c be a map from L into L. We say that c is closure if and only if:

(Def. 14) c is projection and $\mathrm{id}_L \leq c$.

We introduce c is a closure operator as a synonym of c is closure.

Let L be a non empty relational structure. One can verify that every map from L into L which is closure is also projection.

Let L be a non empty reflexive relational structure. One can verify that there exists a map from L into L which is closure.

Let L be a non empty reflexive relational structure. One can verify that id_L is closure.

Let L be a non empty relational structure and let k be a map from L into L. We say that k is kernel if and only if:

(Def. 15) k is projection and $k \leq \mathrm{id}_L$.

We introduce k is a kernel operator as a synonym of k is kernel.

Let L be a non empty relational structure. One can check that every map from L into L which is kernel is also projection.

Let L be a non empty reflexive relational structure. Note that there exists a map from L into L which is kernel.

Let L be a non empty reflexive relational structure. Note that id_L is kernel.

Next we state two propositions:

(30) Let L be a non empty poset, c be a map from L into L, and X be a subset of L. Suppose c is a closure operator and inf X exists in L and $X \subseteq \mathrm{rng}\, c$. Then $\inf X = c(\inf X)$.

(31) Let L be a non empty poset, k be a map from L into L, and X be a subset of L. Suppose k is a kernel operator and sup X exists in L and $X \subseteq \mathrm{rng}\, k$. Then $\sup X = k(\sup X)$.

Let L_1, L_2 be non empty relational structures and let g be a map from L_1 into L_2. The functor g° yields a map from L_1 into $\mathrm{Im}\, g$ and is defined as follows:

(Def. 16) $g^\circ = (\text{the carrier of } \mathrm{Im}\, g) \restriction g$.

The following proposition is true

(32) For all non empty relational structures L_1, L_2 and for every map g from L_1 into L_2 holds $g^\circ = g$.

Let L_1, L_2 be non empty relational structures and let g be a map from L_1 into L_2. One can verify that g° is onto.

The following proposition is true

(33) Let L_1, L_2 be non empty relational structures and g be a map from L_1 into L_2. If g is monotone, then g° is monotone.

Let L_1, L_2 be non empty relational structures and let g be a map from L_1 into L_2. The functor $g_\circ$ yields a map from $\mathrm{Im}\, g$ into L_2 and is defined as follows:

(Def. 17) $g_\circ = \mathrm{id}_{\mathrm{Im}\, g}$.

We now state the proposition

(34) Let L_1, L_2 be non empty relational structures, g be a map from L_1 into L_2, and s be an element of $\mathrm{Im}\, g$. Then $g_\circ(s) = s$.

Let L_1, L_2 be non empty relational structures and let g be a map from L_1 into L_2. Observe that $g_\circ$ is one-to-one and monotone.

Next we state a number of propositions:

(35) For every non empty relational structure L and for every map f from L into L holds $f_\circ \cdot f^\circ = f$.

(36) For every non empty poset L and for every map f from L into L such that f is idempotent holds $f^\circ \cdot f_\circ = \mathrm{id}_{\mathrm{Im}\, f}$.

(37) Let L be a non empty poset and f be a map from L into L. Suppose f is a projection operator. Then there exists a non empty poset T and there exists a map q from L into T and there exists a map i from T into L such that q is monotone and onto and i is monotone and one-to-one and $f = i \cdot q$ and $\mathrm{id}_T = q \cdot i$.

(38) Let L be a non empty poset and f be a map from L into L. Given a non empty poset T and a map q from L into T and a map i from T into L such that q is monotone and i is monotone and $f = i \cdot q$ and $\mathrm{id}_T = q \cdot i$. Then f is a projection operator.

(39) For every non empty poset L and for every map f from L into L such that f is a closure operator holds $\langle f_\circ, f^\circ \rangle$ is Galois.

(40) Let L be a non empty poset and f be a map from L into L. Suppose f is a closure operator. Then there exists a non empty poset S and there exists a map g from S into L and there exists a map d from L into S such that $\langle g, d \rangle$ is Galois and $f = g \cdot d$.

(41) Let L be a non empty poset and f be a map from L into L. Suppose that

(i) f is monotone, and

(ii) there exists a non empty poset S and there exists a map g from S into L and there exists a map d from L into S such that $\langle g, d \rangle$ is Galois and $f = g \cdot d$.

Then f is a closure operator.

(42) For every non empty poset L and for every map f from L into L such that f is a kernel operator holds $\langle f^\circ, f_\circ \rangle$ is Galois.

(43) Let L be a non empty poset and f be a map from L into L. Suppose f is a kernel operator. Then there exists a non empty poset T and there exists a map g from L into T and there exists a map d from T into L such that $\langle g, d \rangle$ is Galois and $f = d \cdot g$.

(44) Let L be a non empty poset and f be a map from L into L. Suppose that

(i) f is monotone, and

(ii) there exists a non empty poset T and there exists a map g from L into T and there exists a map d from T into L such that $\langle g, d \rangle$ is Galois and $f = d \cdot g$.

Then f is a kernel operator.

(45) Let L be a non empty poset and p be a map from L into L. Suppose p is a projection operator. Then $\mathrm{rng}\, p = \{c; c$ ranges over elements of L: $c \leq p(c)\} \cap \{k; k$ ranges over elements of L: $p(k) \leq k\}$.

(46) Let L be a non empty poset and p be a map from L into L. Suppose p is a projection operator. Then

(i) $\{c; c$ ranges over elements of L: $c \leq p(c)\}$ is a non empty subset of L, and

(ii) $\{k; k$ ranges over elements of L: $p(k) \leq k\}$ is a non empty subset of L.

(47) Let L be a non empty poset and p be a map from L into L. Suppose p is a projection operator. Then $\mathrm{rng}(p{\restriction}\{c; c$ ranges over elements of L: $c \leq p(c)\}) = \mathrm{rng}\, p$ and $\mathrm{rng}(p{\restriction}\{k; k$ ranges over elements of L: $p(k) \leq k\}) = \mathrm{rng}\, p$.

(48) Let L be a non empty poset and p be a map from L into L. Suppose p is a projection operator. Let L_4 be a non empty subset of L and L_5 be a non empty subset of L. Suppose $L_4 = \{c; c$ ranges over elements of L: $c \leq p(c)\}$. Then $p{\restriction}L_4$ is a map from $\mathrm{sub}(L_4)$ into $\mathrm{sub}(L_4)$.

(49) Let L be a non empty poset and p be a map from L into L. Suppose p is a projection operator. Let L_5 be a non empty subset of L. Suppose $L_5 = \{k; k$ ranges over elements of L: $p(k) \leq k\}$. Then $p \restriction L_5$ is a map from $\mathrm{sub}(L_5)$ into $\mathrm{sub}(L_5)$.

(50) Let L be a non empty poset and p be a map from L into L. Suppose p is a projection operator. Let L_4 be a non empty subset of L. Suppose $L_4 = \{c; c$ ranges over elements of L: $c \leq p(c)\}$. Let p_1 be a map from $\mathrm{sub}(L_4)$ into $\mathrm{sub}(L_4)$. If $p_1 = p \restriction L_4$, then p_1 is a closure operator.

(51) Let L be a non empty poset and p be a map from L into L. Suppose p is a projection operator. Let L_5 be a non empty subset of L. Suppose $L_5 = \{k; k$ ranges over elements of L: $p(k) \leq k\}$. Let p_2 be a map from $\mathrm{sub}(L_5)$ into $\mathrm{sub}(L_5)$. If $p_2 = p \restriction L_5$, then p_2 is a kernel operator.

(52) Let L be a non empty poset and p be a map from L into L. Suppose p is monotone. Let L_4 be a subset of L. If $L_4 = \{c; c$ ranges over elements of L: $c \leq p(c)\}$, then $\mathrm{sub}(L_4)$ is sups-inheriting.

(53) Let L be a non empty poset and p be a map from L into L. Suppose p is monotone. Let L_5 be a subset of L. If $L_5 = \{k; k$ ranges over elements of L: $p(k) \leq k\}$, then $\mathrm{sub}(L_5)$ is infs-inheriting.

(54) Let L be a non empty poset and p be a map from L into L. Suppose p is a projection operator. Let L_4 be a non empty subset of L such that $L_4 = \{c; c$ ranges over elements of L: $c \leq p(c)\}$. Then

(i) if p is infs-preserving, then $\mathrm{sub}(L_4)$ is infs-inheriting and $\mathrm{Im}\, p$ is infs-inheriting, and

(ii) if p is filtered-infs-preserving, then $\mathrm{sub}(L_4)$ is filtered-infs-inheriting and $\mathrm{Im}\, p$ is filtered-infs-inheriting.

(55) Let L be a non empty poset and p be a map from L into L. Suppose p is a projection operator. Let L_5 be a non empty subset of L such that $L_5 = \{k; k$ ranges over elements of L: $p(k) \leq k\}$. Then

(i) if p is sups-preserving, then $\mathrm{sub}(L_5)$ is sups-inheriting and $\mathrm{Im}\, p$ is sups-inheriting, and

(ii) if p is directed-sups-preserving, then $\mathrm{sub}(L_5)$ is directed-sups-inheriting and $\mathrm{Im}\, p$ is directed-sups-inheriting.

(56) Let L be a non empty poset and p be a map from L into L. Then

(i) if p is a closure operator, then $\mathrm{Im}\, p$ is infs-inheriting, and

(ii) if p is a kernel operator, then $\mathrm{Im}\, p$ is sups-inheriting.

(57) Let L be a complete non empty poset and p be a map from L into L. If p is a projection operator, then $\mathrm{Im}\, p$ is complete.

(58) Let L be a non empty poset and c be a map from L into L. Suppose c is a closure operator. Then

(i) c° is sups-preserving, and

(ii) for every subset X of L such that $X \subseteq$ the carrier of $\mathrm{Im}\, c$ and $\sup X$ exists in L holds $\sup X$ exists in $\mathrm{Im}\, c$ and $\bigsqcup_{\mathrm{Im}\, c} X = c(\bigsqcup_L X)$.

(59) Let L be a non empty poset and k be a map from L into L. Suppose k is a kernel operator. Then

(i) k° is infs-preserving, and

(ii) for every subset X of L such that $X \subseteq$ the carrier of $\mathrm{Im}\, k$ and $\inf X$ exists in L holds $\inf X$ exists in $\mathrm{Im}\, k$ and $\bigsqcap_{\mathrm{Im}\, k} X = k(\bigsqcap_L X)$.

3. HEYTING ALGEBRA

One can prove the following propositions:

(60) For every complete non empty poset L holds $\langle \mathrm{IdsMap}(L), \mathrm{SupMap}(L) \rangle$ is Galois and $\mathrm{SupMap}(L)$ is sups-preserving.

(61) For every complete non empty poset L holds $\mathrm{IdsMap}(L) \cdot \mathrm{SupMap}(L)$ is a closure operator and $\mathrm{Im}(\mathrm{IdsMap}(L) \cdot \mathrm{SupMap}(L))$ and L are isomorphic.

Let S be a non empty relational structure and let x be an element of S. The functor $x \sqcap \square$ yielding a map from S into S is defined by:

(Def. 18) For every element s of S holds $(x \sqcap \square)(s) = x \sqcap s$.

The following two propositions are true:

(62) For every non empty relational structure S and for all elements x, t of S holds $\{s; s$ ranges over elements of S: $x \sqcap s \leq t\} = (x \sqcap \square)^{-1}(\downarrow t)$.

(63) For every semilattice S and for every element x of S holds $x \sqcap \square$ is monotone.

Let S be a semilattice and let x be an element of S. Observe that $x \sqcap \square$ is monotone.
Next we state several propositions:

(64) Let S be a non empty relational structure, x be an element of S, and X be a subset of S. Then $(x \sqcap \square)^{\circ} X = \{x \sqcap y; y$ ranges over elements of S: $y \in X\}$.

(65) Let S be a semilattice. Then for every element x of S holds $x \sqcap \square$ has an upper adjoint if and only if for all elements x, t of S holds $\max \{s; s$ ranges over elements of S: $x \sqcap s \leq t\}$ exists in S.

(66) Let S be a semilattice. Suppose that for every element x of S holds $x \sqcap \square$ has an upper adjoint. Let X be a subset of S. Suppose $\sup X$ exists in S. Let x be an element of S. Then $x \sqcap \bigsqcup_S X = \bigsqcup_S \{x \sqcap y; y$ ranges over elements of S: $y \in X\}$.

(67) Let S be a complete non empty poset. Then for every element x of S holds $x \sqcap \square$ has an upper adjoint if and only if for every subset X of S and for every element x of S holds $x \sqcap \bigsqcup_S X = \bigsqcup_S \{x \sqcap y; y$ ranges over elements of S: $y \in X\}$.

(68) Let S be a lattice. Suppose that for every subset X of S such that $\sup X$ exists in S and for every element x of S holds $x \sqcap \bigsqcup_S X = \bigsqcup_S \{x \sqcap y; y$ ranges over elements of S: $y \in X\}$. Then S is distributive.

Let H be a non empty relational structure. We say that H is Heyting if and only if:

(Def. 19) H is a lattice and for every element x of H holds $x \sqcap \square$ has an upper adjoint.

We introduce H is a Heyting algebra as a synonym of H is Heyting.

Let us mention that every non empty relational structure which is Heyting is also reflexive, transitive, and antisymmetric and has g.l.b.'s and l.u.b.'s.

Let H be a non empty relational structure and let a be an element of H. Let us assume that H is Heyting. The functor $a \Rightarrow \square$ yields a map from H into H and is defined as follows:

(Def. 20) $\langle a \Rightarrow \square, a \sqcap \square \rangle$ is Galois.

The following proposition is true

(69) For every non empty relational structure H such that H is a Heyting algebra holds H is distributive.

Let us mention that every non empty relational structure which is Heyting is also distributive.

Let H be a non empty relational structure and let a, y be elements of H. The functor $a \Rightarrow y$ yielding an element of H is defined as follows:

(Def. 21) $a \Rightarrow y = (a \Rightarrow \Box)(y)$.

One can prove the following propositions:

(70) Let H be a non empty relational structure. Suppose H is a Heyting algebra. Let x, a, y be elements of H. Then $x \geq a \sqcap y$ if and only if $a \Rightarrow x \geq y$.

(71) For every non empty relational structure H such that H is a Heyting algebra holds H is upper-bounded.

Let us observe that every non empty relational structure which is Heyting is also upper-bounded.

Next we state a number of propositions:

(72) Let H be a non empty relational structure. Suppose H is a Heyting algebra. Let a, b be elements of H. Then $\top_H = a \Rightarrow b$ if and only if $a \leq b$.

(73) For every non empty relational structure H such that H is a Heyting algebra and for every element a of H holds $\top_H = a \Rightarrow a$.

(74) Let H be a non empty relational structure. Suppose H is a Heyting algebra. Let a, b be elements of H. If $\top_H = a \Rightarrow b$ and $\top_H = b \Rightarrow a$, then $a = b$.

(75) Let H be a non empty relational structure. If H is a Heyting algebra, then for all elements a, b of H holds $b \leq a \Rightarrow b$.

(76) Let H be a non empty relational structure. If H is a Heyting algebra, then for every element a of H holds $\top_H = a \Rightarrow \top_H$.

(77) For every non empty relational structure H such that H is a Heyting algebra and for every element b of H holds $b = \top_H \Rightarrow b$.

(78) Let H be a non empty relational structure. Suppose H is a Heyting algebra. Let a, b, c be elements of H. If $a \leq b$, then $b \Rightarrow c \leq a \Rightarrow c$.

(79) Let H be a non empty relational structure. Suppose H is a Heyting algebra. Let a, b, c be elements of H. If $b \leq c$, then $a \Rightarrow b \leq a \Rightarrow c$.

(80) Let H be a non empty relational structure. Suppose H is a Heyting algebra. Let a, b be elements of H. Then $a \sqcap (a \Rightarrow b) = a \sqcap b$.

(81) Let H be a non empty relational structure. Suppose H is a Heyting algebra. Let a, b, c be elements of H. Then $a \sqcup b \Rightarrow c = (a \Rightarrow c) \sqcap (b \Rightarrow c)$.

Let H be a non empty relational structure and let a be an element of H. The functor $\neg a$ yields an element of H and is defined as follows:

(Def. 22) $\neg a = a \Rightarrow \bot_H$.

One can prove the following propositions:

(82) Let H be a non empty relational structure. Suppose H is a Heyting algebra and lower-bounded. Let a be an element of H. Then $\neg a$ is a maximum of $\{x; x$ ranges over elements of H: $a \sqcap x = \bot_H\}$.

(83) Let H be a non empty relational structure. If H is a Heyting algebra and lower-bounded, then $\neg(\bot_H) = \top_H$ and $\neg(\top_H) = \bot_H$.

(84) Let H be a non empty lower-bounded relational structure. Suppose H is a Heyting algebra. Let a, b be elements of H. Then $\neg a \geq b$ if and only if $\neg b \geq a$.

(85) Let H be a non empty lower-bounded relational structure. Suppose H is a Heyting algebra. Let a, b be elements of H. Then $\neg a \geq b$ if and only if $a \sqcap b = \bot_H$.

(86) Let H be a non empty lower-bounded relational structure. Suppose H is a Heyting algebra. Let a, b be elements of H. If $a \leq b$, then $\neg b \leq \neg a$.

(87) Let H be a non empty lower-bounded relational structure. Suppose H is a Heyting algebra. Let a, b be elements of H. Then $\neg(a \sqcup b) = \neg a \sqcap \neg b$.

(88) Let H be a non empty lower-bounded relational structure. Suppose H is a Heyting algebra. Let a, b be elements of H. Then $\neg(a \sqcap b) \geq \neg a \sqcup \neg b$.

Let L be a non empty relational structure and let x, y be elements of L. We say that y is a complement of x if and only if:

(Def. 23) $x \sqcup y = \top_L$ and $x \sqcap y = \bot_L$.

Let L be a non empty relational structure. We say that L is complemented if and only if:

(Def. 24) For every element x of L holds there exists an element of L which is a complement of x.

Let X be a set. Note that $2^X_\subseteq$ is complemented.

One can prove the following two propositions:

(89) Let L be a bounded lattice. Suppose L is a Heyting algebra and for every element x of L holds $\neg\neg x = x$. Let x be an element of L. Then $\neg x$ is a complement of x.

(90) Let L be a bounded lattice. Then L is distributive and complemented if and only if L is a Heyting algebra and for every element x of L holds $\neg\neg x = x$.

Let B be a non empty relational structure. We say that B is Boolean if and only if:

(Def. 25) B is a lattice, bounded, distributive, and complemented.

We introduce B is a Boolean algebra and B is a Boolean lattice as synonyms of B is Boolean.

Let us note that every non empty relational structure which is Boolean is also reflexive, transitive, antisymmetric, bounded, distributive, and complemented and has g.l.b.'s and l.u.b.'s.

Let us note that every non empty relational structure which is reflexive, transitive, antisymmetric, bounded, distributive, and complemented and has g.l.b.'s and l.u.b.'s is also Boolean.

Let us note that every non empty relational structure which is Boolean is also Heyting.

Let us observe that there exists a lattice which is strict, Boolean, and non empty.

Let us observe that there exists a lattice which is strict, Heyting, and non empty.

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