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# Function Spaces in the Category of Directed Suprema Preserving Maps[1]

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The articles [28], [13], [34], [30], [29], [35], [10], [12], [9], [14], [1], [2], [23], [31], [4], [24], [27], [26], [11], [33], [3], [5], [6], [18], [36], [15], [20], [7], [25], [21], [32], [17], [19], [22], and [8] provide the notation and terminology for this paper.

## 1. Currying, Uncurrying and Commuting Functions

Let $F$ be a function. We say that $F$ is uncurrying if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) For every set $x$ such that $x \in \mathrm{dom}\, F$ holds $x$ is a function yielding function, and

(ii) for every function $f$ such that $f \in \mathrm{dom}\, F$ holds $F(f) = \mathrm{uncurry}\, f$.

We say that $F$ is currying if and only if the conditions (Def. 2) are satisfied.

(Def. 2)(i) For every set $x$ such that $x \in \mathrm{dom}\, F$ holds $x$ is a function and $\pi_1(x)$ is a binary relation, and

(ii) for every function $f$ such that $f \in \mathrm{dom}\, F$ holds $F(f) = \mathrm{curry}\, f$.

We say that $F$ is commuting if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) For every set $x$ such that $x \in \mathrm{dom}\, F$ holds $x$ is a function yielding function, and

(ii) for every function $f$ such that $f \in \mathrm{dom}\, F$ holds $F(f) = \mathrm{commute}(f)$.

Let us observe that every function which is empty is also uncurrying, currying, and commuting.

Let us note that there exists a function which is uncurrying, currying, and commuting.

Let $F$ be an uncurrying function and let $X$ be a set. One can check that $F \restriction X$ is uncurrying.

Let $F$ be a currying function and let $X$ be a set. Note that $F \restriction X$ is currying.

Next we state two propositions:

(1) Let $X$, $Y$, $Z$, $D$ be sets. Suppose $D \subseteq (Z^Y)^X$. Then there exists a many sorted set $F$ indexed by $D$ such that $F$ is uncurrying and $\mathrm{rng}\, F \subseteq Z^{[:X,Y:]}$.

(2) Let $X$, $Y$, $Z$, $D$ be sets. Suppose $D \subseteq Z^{[:X,Y:]}$. Then there exists a many sorted set $F$ indexed by $D$ such that $F$ is currying and if if $Y = \emptyset$, then $X = \emptyset$, then $\mathrm{rng}\, F \subseteq (Z^Y)^X$.

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Let $X$, $Y$, $Z$ be sets. One can verify that there exists a many sorted set indexed by $(Z^Y)^X$ which is uncurrying and there exists a many sorted set indexed by $Z^{[:X,Y:]}$ which is currying.

The following propositions are true:

(3) Let $A$, $B$ be non empty sets, $C$ be a set, and $f$, $g$ be commuting functions. If $\mathrm{dom}\, f \subseteq (C^B)^A$ and $\mathrm{rng}\, f \subseteq \mathrm{dom}\, g$, then $g \cdot f = \mathrm{id}_{\mathrm{dom}\, f}$.

(4) Let $B$ be a non empty set, $A$, $C$ be sets, $f$ be an uncurrying function, and $g$ be a currying function. If $\mathrm{dom}\, f \subseteq (C^B)^A$ and $\mathrm{rng}\, f \subseteq \mathrm{dom}\, g$, then $g \cdot f = \mathrm{id}_{\mathrm{dom}\, f}$.

(5) Let $A$, $B$, $C$ be sets, $f$ be a currying function, and $g$ be an uncurrying function. If $\mathrm{dom}\, f \subseteq C^{[:A,B:]}$ and $\mathrm{rng}\, f \subseteq \mathrm{dom}\, g$, then $g \cdot f = \mathrm{id}_{\mathrm{dom}\, f}$.

(6) For every function yielding function $f$ and for all sets $i$, $A$ such that $i \in \mathrm{dom}\,\mathrm{commute}(f)$ holds $(\mathrm{commute}(f))(i)^{\circ}A \subseteq \pi_i f^{\circ}A$.

(7) Let $f$ be a function yielding function and $i$, $A$ be sets. If for every function $g$ such that $g \in f^{\circ}A$ holds $i \in \mathrm{dom}\, g$, then $\pi_i f^{\circ}A \subseteq (\mathrm{commute}(f))(i)^{\circ}A$.

(8) For all sets $X$, $Y$ and for every function $f$ such that $\mathrm{rng}\, f \subseteq Y^X$ and for all sets $i$, $A$ such that $i \in X$ holds $(\mathrm{commute}(f))(i)^{\circ}A = \pi_i f^{\circ}A$.

(9) For every function $f$ and for all sets $i$, $A$ such that $[:A, \{i\}:] \subseteq \mathrm{dom}\, f$ holds $\pi_i(\mathrm{curry}\, f)^{\circ}A = f^{\circ}[:A, \{i\}:]$.

Let $X$ be a set and let $Y$ be a non empty functional set. Note that every function from $X$ into $Y$ is function yielding.

Let $T$ be a constituted functions 1-sorted structure. One can verify that the carrier of $T$ is functional.

Let $X$ be a set and let $L$ be a non empty relational structure. Note that $L^X$ is constituted functions.

Let us note that there exists a lattice which is constituted functions, complete, and strict and there exists a 1-sorted structure which is constituted functions and non empty.

Let $T$ be a constituted functions non empty relational structure. Note that every non empty relational substructure of $T$ is constituted functions.

We now state four propositions:

(10) Let $S$, $T$ be complete lattices, $f$ be an idempotent map from $T$ into $T$, and $h$ be a map from $S$ into $\mathrm{Im}\, f$. Then $f \cdot h = h$.

(11) Let $S$ be a non empty relational structure and $T$, $T_1$ be non empty relational structures. Suppose $T$ is a relational substructure of $T_1$. Let $f$ be a map from $S$ into $T$ and $f_1$ be a map from $S$ into $T_1$. If $f$ is monotone and $f = f_1$, then $f_1$ is monotone.

(12) Let $S$ be a non empty relational structure and $T$, $T_1$ be non empty relational structures. Suppose $T$ is a full relational substructure of $T_1$. Let $f$ be a map from $S$ into $T$ and $f_1$ be a map from $S$ into $T_1$. If $f_1$ is monotone and $f = f_1$, then $f$ is monotone.

(13) For every set $X$ and for every subset $V$ of $X$ holds $(\chi_{V,X})^{-1}(\{1\}) = V$ and $(\chi_{V,X})^{-1}(\{0\}) = X \setminus V$.

## 2. Maps of Power Posets

Let $X$ be a non empty set, let $T$ be a non empty relational structure, let $f$ be an element of $T^X$, and let $x$ be an element of $X$. Then $f(x)$ is an element of $T$.

Next we state several propositions:

(14) Let $X$ be a non empty set, $T$ be a non empty relational structure, and $f$, $g$ be elements of $T^X$. Then $f \leq g$ if and only if for every element $x$ of $X$ holds $f(x) \leq g(x)$.

(15) Let $X$ be a set and $L$, $S$ be non empty relational structures. Suppose the relational structure of $L =$ the relational structure of $S$. Then $L^X = S^X$.

(16) Let $S_1$, $S_2$, $T_1$, $T_2$ be non empty topological spaces. Suppose that

(i) the topological structure of $S_1 =$ the topological structure of $S_2$, and

(ii) the topological structure of $T_1 =$ the topological structure of $T_2$.

Then $[S_1 \to T_1] = [S_2 \to T_2]$.

(17) Let $X$ be a set. Then there exists a map $f$ from $2^X_\subseteq$ into $(2^1_\subseteq)^X$ such that $f$ is isomorphic and for every subset $Y$ of $X$ holds $f(Y) = \chi_{Y,X}$.

(18) For every set $X$ holds $2^X_\subseteq$ and $(2^1_\subseteq)^X$ are isomorphic.

(19) Let $X$, $Y$ be non empty sets, $T$ be a non empty poset, $S_1$ be a full non empty relational substructure of $(T^X)^Y$, $S_2$ be a full non empty relational substructure of $(T^Y)^X$, and $F$ be a map from $S_1$ into $S_2$. If $F$ is commuting, then $F$ is monotone.

(20) Let $X$, $Y$ be non empty sets, $T$ be a non empty poset, $S_1$ be a full non empty relational substructure of $(T^Y)^X$, $S_2$ be a full non empty relational substructure of $T^{[:X,Y:]}$, and $F$ be a map from $S_1$ into $S_2$. If $F$ is uncurrying, then $F$ is monotone.

(21) Let $X$, $Y$ be non empty sets, $T$ be a non empty poset, $S_1$ be a full non empty relational substructure of $(T^Y)^X$, $S_2$ be a full non empty relational substructure of $T^{[:X,Y:]}$, and $F$ be a map from $S_2$ into $S_1$. If $F$ is currying, then $F$ is monotone.

## 3. Posets of Directed Suprema Preserving Maps

Let $S$ be a non empty relational structure and let $T$ be a non empty reflexive antisymmetric relational structure. The functor $\mathrm{UPS}(S,T)$ yields a strict relational structure and is defined by the conditions (Def. 4).

(Def. 4)(i) $\mathrm{UPS}(S,T)$ is a full relational substructure of $T^{\text{the carrier of } S}$, and

(ii) for every set $x$ holds $x \in$ the carrier of $\mathrm{UPS}(S,T)$ iff $x$ is a directed-sups-preserving map from $S$ into $T$.

Let $S$ be a non empty relational structure and let $T$ be a non empty reflexive antisymmetric relational structure. One can check that $\mathrm{UPS}(S,T)$ is non empty, reflexive, antisymmetric, and constituted functions.

Let $S$ be a non empty relational structure and let $T$ be a non empty poset. Note that $\mathrm{UPS}(S,T)$ is transitive.

We now state the proposition

(22) Let $S$ be a non empty relational structure and $T$ be a non empty reflexive antisymmetric relational structure. Then the carrier of $\mathrm{UPS}(S,T) \subseteq$ (the carrier of $T$)$^{\text{the carrier of } S}$.

Let $S$ be a non empty relational structure, let $T$ be a non empty reflexive antisymmetric relational structure, let $f$ be an element of $\mathrm{UPS}(S,T)$, and let $s$ be an element of $S$. Then $f(s)$ is an element of $T$.

One can prove the following propositions:

(23) Let $S$ be a non empty relational structure, $T$ be a non empty reflexive antisymmetric relational structure, and $f$, $g$ be elements of $\mathrm{UPS}(S,T)$. Then $f \leq g$ if and only if for every element $s$ of $S$ holds $f(s) \leq g(s)$.

(24) For all complete Scott top-lattices $S$, $T$ holds $\mathrm{UPS}(S,T) = \mathrm{SCMaps}(S,T)$.

(25) Let $S$, $S'$ be non empty relational structures and $T$, $T'$ be non empty reflexive antisymmetric relational structures. Suppose that

(i) the relational structure of $S =$ the relational structure of $S'$, and

(ii) the relational structure of $T =$ the relational structure of $T'$.

Then $\mathrm{UPS}(S,T) = \mathrm{UPS}(S',T')$.

Let $S$, $T$ be complete lattices. Note that $\mathrm{UPS}(S,T)$ is complete.
One can prove the following two propositions:

(26) Let $S$, $T$ be complete lattices. Then $\mathrm{UPS}(S,T)$ is a sups-inheriting relational substructure of $T^{\text{the carrier of } S}$.

(27) For all complete lattices $S$, $T$ and for every subset $A$ of $\mathrm{UPS}(S,T)$ holds $\sup A = \bigsqcup_{(T^{\text{the carrier of } S})} A$.

Let $S_1$, $S_2$, $T_1$, $T_2$ be non empty reflexive antisymmetric relational structures and let $f$ be a map from $S_1$ into $S_2$. Let us assume that $f$ is directed-sups-preserving. Let $g$ be a map from $T_1$ into $T_2$. Let us assume that $g$ is directed-sups-preserving. The functor $\mathrm{UPS}(f,g)$ yields a map from $\mathrm{UPS}(S_2,T_1)$ into $\mathrm{UPS}(S_1,T_2)$ and is defined by:

(Def. 5) For every directed-sups-preserving map $h$ from $S_2$ into $T_1$ holds $(\mathrm{UPS}(f,g))(h) = g \cdot h \cdot f$.

One can prove the following propositions:

(28) Let $S_1$, $S_2$, $S_3$, $T_1$, $T_2$, $T_3$ be non empty posets, $f_1$ be a directed-sups-preserving map from $S_2$ into $S_3$, $f_2$ be a directed-sups-preserving map from $S_1$ into $S_2$, $g_1$ be a directed-sups-preserving map from $T_1$ into $T_2$, and $g_2$ be a directed-sups-preserving map from $T_2$ into $T_3$. Then $\mathrm{UPS}(f_2,g_2) \cdot \mathrm{UPS}(f_1,g_1) = \mathrm{UPS}(f_1 \cdot f_2, g_2 \cdot g_1)$.

(29) For all non empty reflexive antisymmetric relational structures $S$, $T$ holds $\mathrm{UPS}(\mathrm{id}_S, \mathrm{id}_T) = \mathrm{id}_{\mathrm{UPS}(S,T)}$.

(30) Let $S_1$, $S_2$, $T_1$, $T_2$ be complete lattices, $f$ be a directed-sups-preserving map from $S_1$ into $S_2$, and $g$ be a directed-sups-preserving map from $T_1$ into $T_2$. Then $\mathrm{UPS}(f,g)$ is directed-sups-preserving.

(31) $\Omega$(the Sierpiński space) is Scott.

(32) For every complete Scott top-lattice $S$ holds $[S \to \text{the Sierpiński space}] = \mathrm{UPS}(S, 2^1_\subseteq)$.

(33) Let $S$ be a complete lattice. Then there exists a map $F$ from $\mathrm{UPS}(S, 2^1_\subseteq)$ into $\langle \sigma(S), \subseteq \rangle$ such that $F$ is isomorphic and for every directed-sups-preserving map $f$ from $S$ into $2^1_\subseteq$ holds $F(f) = f^{-1}(\{1\})$.

(34) For every complete lattice $S$ holds $\mathrm{UPS}(S, 2^1_\subseteq)$ and $\langle \sigma(S), \subseteq \rangle$ are isomorphic.

(35) Let $S_1$, $S_2$, $T_1$, $T_2$ be complete lattices, $f$ be a map from $S_1$ into $S_2$, and $g$ be a map from $T_1$ into $T_2$. If $f$ is isomorphic and $g$ is isomorphic, then $\mathrm{UPS}(f,g)$ is isomorphic.

(36) Let $S_1$, $S_2$, $T_1$, $T_2$ be complete lattices. Suppose $S_1$ and $S_2$ are isomorphic and $T_1$ and $T_2$ are isomorphic. Then $\mathrm{UPS}(S_2,T_1)$ and $\mathrm{UPS}(S_1,T_2)$ are isomorphic.

(37) Let $S$, $T$ be complete lattices and $f$ be a directed-sups-preserving projection map from $T$ into $T$. Then $\mathrm{Im}\,\mathrm{UPS}(\mathrm{id}_S, f) = \mathrm{UPS}(S, \mathrm{Im} f)$.

(38) Let $X$ be a non empty set, $S$, $T$ be non empty posets, $f$ be a directed-sups-preserving map from $S$ into $T^X$, and $i$ be an element of $X$. Then $(\mathrm{commute}(f))(i)$ is a directed-sups-preserving map from $S$ into $T$.

(39) Let $X$ be a non empty set, $S$, $T$ be non empty posets, and $f$ be a directed-sups-preserving map from $S$ into $T^X$. Then $\mathrm{commute}(f)$ is a function from $X$ into the carrier of $\mathrm{UPS}(S,T)$.

(40) Let $X$ be a non empty set, $S$, $T$ be non empty posets, and $f$ be a function from $X$ into the carrier of $\mathrm{UPS}(S,T)$. Then $\mathrm{commute}(f)$ is a directed-sups-preserving map from $S$ into $T^X$.

(41) For every non empty set $X$ and for all non empty posets $S$, $T$ holds there exists a map from $\mathrm{UPS}(S,T^X)$ into $\mathrm{UPS}(S,T)^X$ which is commuting and isomorphic.

(42) For every non empty set $X$ and for all non empty posets $S$, $T$ holds $\mathrm{UPS}(S,T^X)$ and $(\mathrm{UPS}(S,T))^X$ are isomorphic.

(43) For all continuous complete lattices $S$, $T$ holds $\mathrm{UPS}(S,T)$ is continuous.

(44) For all algebraic complete lattices $S$, $T$ holds $\mathrm{UPS}(S,T)$ is algebraic.

(45) Let $R$, $S$, $T$ be complete lattices and $f$ be a directed-sups-preserving map from $R$ into $\mathrm{UPS}(S,T)$. Then uncurry $f$ is a directed-sups-preserving map from $[{:}\,R, S\,{:}]$ into $T$.

(46) Let $R$, $S$, $T$ be complete lattices and $f$ be a directed-sups-preserving map from $[{:}\,R, S\,{:}]$ into $T$. Then curry $f$ is a directed-sups-preserving map from $R$ into $\mathrm{UPS}(S,T)$.

(47) For all complete lattices $R$, $S$, $T$ holds there exists a map from $\mathrm{UPS}(R,\mathrm{UPS}(S,T))$ into $\mathrm{UPS}([{:}\,R, S\,{:}],T)$ which is uncurrying and isomorphic.

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