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The Lawson Topology[1]

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Summary. The article includes definitions, lemmas and theorems 1.1–1.7, 1.9, 1.10 presented in Chapter III of [11, pp. 142–146].

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The articles [19], [8], [24], [16], [25], [6], [10], [7], [23], [1], [17], [20], [9], [2], [3], [12], [18], [13], [14], [21], [4], [15], [22], and [5] provide the notation and terminology for this paper.

1. Lower Topology

Let T be a non empty FR-structure. We say that T is lower if and only if:

(Def. 1) $\{(\uparrow x)^{\mathrm{c}} : x$ ranges over elements of $T\}$ is a prebasis of T.

Let us note that every non empty reflexive topological space-like FR-structure which is trivial is also lower.

Let us note that there exists a top-lattice which is lower, trivial, complete, and strict.

One can prove the following proposition

(1) For every non empty relational structure L_1 holds there exists a strict correct topological augmentation of L_1 which is lower.

Let R be a non empty relational structure. Note that there exists a strict correct topological augmentation of R which is lower.

We now state the proposition

(2) Let L_2, L_3 be topological space-like lower non empty FR-structures. Suppose the relational structure of $L_2 =$ the relational structure of L_3. Then the topology of $L_2 =$ the topology of L_3.

Let R be a non empty relational structure. The functor $\omega(R)$ yielding a family of subsets of R is defined by:

(Def. 2) For every lower correct topological augmentation T of R holds $\omega(R) =$ the topology of T.

One can prove the following propositions:

(3) Let R_1, R_2 be non empty relational structures. Suppose the relational structure of $R_1 =$ the relational structure of R_2. Then $\omega(R_1) = \omega(R_2)$.

(4) For every lower non empty FR-structure T and for every point x of T holds $(\uparrow x)^{\mathrm{c}}$ is open and $\uparrow x$ is closed.

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(5) For every transitive lower non empty FR-structure T and for every subset A of T such that A is open holds A is lower.

(6) For every transitive lower non empty FR-structure T and for every subset A of T such that A is closed holds A is upper.

(7) Let T be a non empty topological space-like FR-structure. Then T is lower if and only if $\{(\uparrow F)^{c}; F$ ranges over subsets of T: F is finite$\}$ is a basis of T.

(8) Let S, T be lower complete top-lattices and f be a map from S into T. Suppose that for every non empty subset X of S holds f preserves inf of X. Then f is continuous.

(9) Let S, T be lower complete top-lattices and f be a map from S into T. If f is infs-preserving, then f is continuous.

(10) Let T be a lower complete top-lattice, B_1 be a prebasis of T, and F be a non empty filtered subset of T. Suppose that for every subset A of T such that $A \in B_1$ and $\inf F \in A$ holds F meets A. Then $\inf F \in \overline{F}$.

(11) Let S, T be lower complete top-lattices and f be a map from S into T. If f is continuous, then f is filtered-infs-preserving.

(12) Let S, T be lower complete top-lattices and f be a map from S into T. Suppose f is continuous and for every finite subset X of S holds f preserves inf of X. Then f is infs-preserving.

(13) Let T be a lower topological space-like reflexive transitive non empty FR-structure and x be a point of T. Then $\overline{\{x\}} = \uparrow x$.

A top-poset is a topological space-like reflexive transitive antisymmetric FR-structure.

One can check that every non empty top-poset which is lower is also T_0.

Let R be a lower-bounded non empty relational structure. Observe that every topological augmentation of R is lower-bounded.

Next we state four propositions:

(14) Let S, T be non empty relational structures, s be an element of S, and t be an element of T. Then $(\uparrow\langle s, t\rangle)^{c} = [: (\uparrow s)^{c}$, the carrier of $T :] \cup [:$ the carrier of S, $(\uparrow t)^{c} :]$.

(15) Let S, T be lower-bounded non empty posets, S' be a lower correct topological augmentation of S, and T' be a lower correct topological augmentation of T. Then $\omega([: S, T :]) =$ the topology of $[: S'$, (T' **qua** non empty topological space) $:]$.

(16) Let S, T be lower lower-bounded non empty top-posets. Then $\omega([: S$, (T **qua** poset) $:]) =$ the topology of $[: S$, (T **qua** non empty topological space) $:]$.

(17) Let T, T_2 be lower complete top-lattices. Suppose T_2 is a topological augmentation of $[: T$, (T **qua** lattice) $:]$. Let f be a map from T_2 into T. If $f = \sqcap_T$, then f is continuous.

2. Refinements Revisited

The scheme *TopInd* deals with a top-lattice $\mathcal{A}$ and a unary predicate $\mathcal{P}$, and states that:

For every subset A of $\mathcal{A}$ such that A is open holds $\mathcal{P}[A]$

provided the parameters meet the following conditions:

- There exists a prebasis K of $\mathcal{A}$ such that for every subset A of $\mathcal{A}$ such that $A \in K$ holds $\mathcal{P}[A]$,
- For every family F of subsets of $\mathcal{A}$ such that for every subset A of $\mathcal{A}$ such that $A \in F$ holds $\mathcal{P}[A]$ holds $\mathcal{P}[\bigcup F]$,
- For all subsets A_1, A_2 of $\mathcal{A}$ such that $\mathcal{P}[A_1]$ and $\mathcal{P}[A_2]$ holds $\mathcal{P}[A_1 \cap A_2]$, and
- $\mathcal{P}[\Omega_{\mathcal{A}}]$.

The following proposition is true

(18) Let L_2, L_3 be up-complete antisymmetric non empty reflexive relational structures. Suppose that

(i) the relational structure of L_2 = the relational structure of L_3, and

(ii) for every element x of L_2 holds $\downdownarrows x$ is directed and non empty.

If L_2 satisfies axiom of approximation, then L_3 satisfies axiom of approximation.

Let T be a continuous non empty poset. Note that every topological augmentation of T is continuous.

Next we state a number of propositions:

(19) Let T, S be topological spaces, R be a refinement of T and S, and W be a subset of R. If $W \in$ the topology of T or $W \in$ the topology of S, then W is open.

(20) Let T, S be topological spaces, R be a refinement of T and S, V be a subset of T, and W be a subset of R. If $W = V$, then if V is open, then W is open.

(21) Let T, S be topological spaces. Suppose the carrier of T = the carrier of S. Let R be a refinement of T and S, V be a subset of T, and W be a subset of R. If $W = V$, then if V is closed, then W is closed.

(22) Let T be a non empty topological space and K, O be sets such that $K \subseteq O$ and $O \subseteq$ the topology of T. Then

(i) if K is a basis of T, then O is a basis of T, and

(ii) if K is a prebasis of T, then O is a prebasis of T.

(23) Let T_1, T_2 be non empty topological spaces. Suppose the carrier of T_1 = the carrier of T_2. Let T be a refinement of T_1 and T_2, B_2 be a prebasis of T_1, and B_3 be a prebasis of T_2. Then $B_2 \cup B_3$ is a prebasis of T.

(24) Let T_1, S_1, T_2, S_2 be non empty topological spaces, R_1 be a refinement of T_1 and S_1, R_2 be a refinement of T_2 and S_2, f be a map from T_1 into T_2, g be a map from S_1 into S_2, and h be a map from R_1 into R_2. Suppose $h = f$ and $h = g$. If f is continuous and g is continuous, then h is continuous.

(25) Let T be a non empty topological space, K be a prebasis of T, N be a net in T, and p be a point of T. Suppose that for every subset A of T such that $p \in A$ and $A \in K$ holds N is eventually in A. Then $p \in \mathrm{Lim} N$.

(26) Let T be a non empty topological space, N be a net in T, and S be a subset of T. If N is eventually in S, then $\mathrm{Lim} N \subseteq \overline{S}$.

(27) Let R be a non empty relational structure and X be a non empty subset of R. Then the mapping of $\langle X; \mathrm{id} \rangle = \mathrm{id}_X$ and the mapping of $\langle X^{\mathrm{op}}; \mathrm{id} \rangle = \mathrm{id}_X$.

(28) For every reflexive antisymmetric non empty relational structure R and for every element x of R holds $\uparrow x \cap \downarrow x = \{x\}$.

3. Lawson Topology

Let T be a reflexive non empty FR-structure. We say that T is Lawson if and only if:

(Def. 3) $\omega(T) \cup \sigma(T)$ is a prebasis of T.

Next we state the proposition

(29) Let R be a complete lattice, L_1 be a lower correct topological augmentation of R, S be a Scott topological augmentation of R, and T be a correct topological augmentation of R. Then T is Lawson if and only if T is a refinement of S and L_1.

Let R be a complete lattice. One can check that there exists a topological augmentation of R which is Lawson, strict, and correct.

Let us note that there exists a top-lattice which is Scott, complete, and strict and there exists a complete strict top-lattice which is Lawson and continuous.

The following three propositions are true:

(30) For every Lawson complete top-lattice T holds $\sigma(T) \cup \{(\uparrow x)^{\rm c} : x$ ranges over elements of $T\}$ is a prebasis of T.

(31) Let T be a Lawson complete top-lattice. Then $\sigma(T) \cup \{W \setminus \uparrow x; W$ ranges over subsets of T, x ranges over elements of T: $W \in \sigma(T)\}$ is a prebasis of T.

(32) Let T be a Lawson complete top-lattice. Then $\{W \setminus \uparrow F; W$ ranges over subsets of T, F ranges over subsets of T: $W \in \sigma(T) \wedge F$is finite$\}$ is a basis of T.

Let T be a complete lattice. The functor $\lambda(T)$ yielding a family of subsets of T is defined by:

(Def. 4) For every Lawson correct topological augmentation S of T holds $\lambda(T) =$ the topology of S.

Next we state a number of propositions:

(33) For every complete lattice R holds $\lambda(R) = \mathrm{UniCl}(\mathrm{FinMeetCl}(\sigma(R) \cup \omega(R)))$.

(34) Let R be a complete lattice, T be a lower correct topological augmentation of R, S be a Scott correct topological augmentation of R, and M be a refinement of S and T. Then $\lambda(R) =$ the topology of M.

(35) For every lower up-complete top-lattice T and for every subset A of T such that A is open holds A has the property (S).

(36) For every Lawson complete top-lattice T and for every subset A of T such that A is open holds A has the property (S).

(37) Let S be a Scott complete top-lattice, T be a Lawson correct topological augmentation of S, and A be a subset of S. If A is open, then for every subset C of T such that $C = A$ holds C is open.

(38) Let T be a Lawson complete top-lattice and x be an element of T. Then $\uparrow x$ is closed and $\downarrow x$ is closed and $\{x\}$ is closed.

(39) For every Lawson complete top-lattice T and for every element x of T holds $(\uparrow x)^{\rm c}$ is open and $(\downarrow x)^{\rm c}$ is open and $\{x\}^{\rm c}$ is open.

(40) For every Lawson complete continuous top-lattice T and for every element x of T holds $\twoheaduparrow x$ is open and $(\twoheaduparrow x)^{\rm c}$ is closed.

(41) Let S be a Scott complete top-lattice, T be a Lawson correct topological augmentation of S, and A be an upper subset of T. If A is open, then for every subset C of S such that $C = A$ holds C is open.

(42) Let T be a Lawson complete top-lattice and A be a lower subset of T. Then A is closed if and only if A is closed under directed sups.

(43) For every Lawson complete top-lattice T and for every non empty filtered subset F of T holds $\mathrm{Lim}\langle F^{\rm op}; \mathrm{id}\rangle = \{\inf F\}$.

Let us note that every complete top-lattice which is Lawson is also T_1 and compact.

Let us observe that every complete continuous top-lattice which is Lawson is also Hausdorff.

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