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Injective Spaces[1]

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The articles [23], [22], [11], [28], [19], [29], [31], [9], [10], [20], [8], [2], [7], [25], [1], [17], [27], [15], [18], [21], [13], [14], [12], [24], [3], [30], [6], [4], [5], [16], and [26] provide the notation and terminology for this paper.

1. Product Topologies

One can prove the following propositions:

(1) Let x, y, z, Z be sets. Then $Z \subseteq \{x,y,z\}$ if and only if one of the following conditions is satisfied:
$Z = \emptyset$ or $Z = \{x\}$ or $Z = \{y\}$ or $Z = \{z\}$ or $Z = \{x,y\}$ or $Z = \{y,z\}$ or $Z = \{x,z\}$ or $Z = \{x,y,z\}$.

(2) For every set X and for all families A, B of subsets of X such that $B = A \setminus \{\emptyset\}$ or $A = B \cup \{\emptyset\}$ holds $\mathrm{UniCl}(A) = \mathrm{UniCl}(B)$.

(3) Let T be a topological space and K be a family of subsets of T. Then K is a basis of T if and only if $K \setminus \{\emptyset\}$ is a basis of T.

Let F be a binary relation. We say that F is topological space yielding if and only if:

(Def. 1) For every set x such that $x \in \mathrm{rng}\, F$ holds x is a topological structure.

Let us note that every function which is topological space yielding is also 1-sorted yielding.

Let I be a set. One can check that there exists a many sorted set indexed by I which is topological space yielding.

Let I be a set. Observe that there exists a many sorted set indexed by I which is topological space yielding and nonempty.

Let J be a non empty set, let A be a topological space yielding many sorted set indexed by J, and let j be an element of J. Then $A(j)$ is a topological structure.

Let I be a set and let J be a topological space yielding many sorted set indexed by I. The product prebasis for J is a family of subsets of $\prod$(the support of J) and is defined by the condition (Def. 2).

(Def. 2) Let x be a subset of $\prod$(the support of J). Then $x \in$ the product prebasis for J if and only if there exists a set i and there exists a topological structure T and there exists a subset V of T such that $i \in I$ and V is open and $T = J(i)$ and $x = \prod((\text{the support of } J) +\cdot (i,V))$.

Next we state the proposition

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(4) For every set X and for every family A of subsets of X holds $\langle X, \mathrm{UniCl}(\mathrm{FinMeetCl}(A))\rangle$ is topological space-like.

Let I be a set and let J be a topological space yielding nonempty many sorted set indexed by I. The functor $\prod J$ yields a strict topological space and is defined by:

(Def. 3) The carrier of $\prod J = \prod$(the support of J) and the product prebasis for J is a prebasis of $\prod J$.

Let I be a set and let J be a topological space yielding nonempty many sorted set indexed by I. One can check that $\prod J$ is non empty.

Let I be a non empty set, let J be a topological space yielding nonempty many sorted set indexed by I, and let i be an element of I. Then $J(i)$ is a non empty topological structure.

Let I be a set and let J be a topological space yielding nonempty many sorted set indexed by I. One can check that every element of $\prod J$ is function-like and relation-like.

Let I be a non empty set, let J be a topological space yielding nonempty many sorted set indexed by I, let x be an element of $\prod J$, and let i be an element of I. Then $x(i)$ is an element of $J(i)$.

Let I be a non empty set, let J be a topological space yielding nonempty many sorted set indexed by I, and let i be an element of I. The functor $\mathrm{proj}(J,i)$ yields a map from $\prod J$ into $J(i)$ and is defined by:

(Def. 4) $\mathrm{proj}(J,i) = \mathrm{proj}$(the support of J, i).

Next we state three propositions:

(5) Let I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, i be an element of I, and P be a subset of $J(i)$. Then $(\mathrm{proj}(J,i))^{-1}(P) = \prod$((the support of $J) +\cdot (i,P))$.

(6) Let I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, and i be an element of I. Then $\mathrm{proj}(J,i)$ is continuous.

(7) Let X be a non empty topological space, I be a non empty set, J be a topological space yielding nonempty many sorted set indexed by I, and f be a map from X into $\prod J$. Then f is continuous if and only if for every element i of I holds $\mathrm{proj}(J,i) \cdot f$ is continuous.

2. Injective Spaces

Let Z be a topological structure. We say that Z is injective if and only if the condition (Def. 5) is satisfied.

(Def. 5) Let X be a non empty topological space and f be a map from X into Z. Suppose f is continuous. Let Y be a non empty topological space. Suppose X is a subspace of Y. Then there exists a map g from Y into Z such that g is continuous and $g\restriction$the carrier of $X = f$.

The following two propositions are true:

(8) Let I be a non empty set and J be a topological space yielding nonempty many sorted set indexed by I. If for every element i of I holds $J(i)$ is injective, then $\prod J$ is injective.

(9) Let T be a non empty topological space. Suppose T is injective. Let S be a non empty subspace of T. If S is a retract of T, then S is injective.

Let X be a 1-sorted structure, let Y be a topological structure, and let f be a map from X into Y. The functor $\mathrm{Im} f$ yielding a subspace of Y is defined by:

(Def. 6) $\mathrm{Im} f = Y \restriction \mathrm{rng} f$.

Let X be a non empty 1-sorted structure, let Y be a non empty topological structure, and let f be a map from X into Y. One can verify that $\mathrm{Im} f$ is non empty.

Next we state the proposition

(10) Let X be a 1-sorted structure, Y be a topological structure, and f be a map from X into Y. Then the carrier of $\operatorname{Im} f = \operatorname{rng} f$.

Let X be a 1-sorted structure, let Y be a non empty topological structure, and let f be a map from X into Y. The functor f° yielding a map from X into $\operatorname{Im} f$ is defined by:

(Def. 7) $f^\circ = f$.

One can prove the following proposition

(11) Let X, Y be non empty topological spaces and f be a map from X into Y. If f is continuous, then f° is continuous.

Let X be a 1-sorted structure, let Y be a non empty topological structure, and let f be a map from X into Y. Observe that f° is onto.

Let X, Y be topological structures. We say that X is a topological retract of Y if and only if:

(Def. 8) There exists a map f from Y into Y such that f is continuous and $f \cdot f = f$ and $\operatorname{Im} f$ and X are homeomorphic.

Next we state the proposition

(12) Let T, S be non empty topological spaces. Suppose T is injective. Let f be a map from T into S. If f° is a homeomorphism, then T is a topological retract of S.

The Sierpiński space is a strict topological structure and is defined by the conditions (Def. 9).

(Def. 9)(i) The carrier of the Sierpiński space $= \{0,1\}$, and

(ii) the topology of the Sierpiński space $= \{\emptyset, \{1\}, \{0,1\}\}$.

One can check that the Sierpiński space is non empty and topological space-like.

Let us mention that the Sierpiński space is discernible.

Let us note that the Sierpiński space is injective.

Let I be a set and let S be a non empty 1-sorted structure. One can verify that $I \longmapsto S$ is nonempty.

Let I be a set and let T be a topological structure. Observe that $I \longmapsto T$ is topological space yielding.

Let I be a set and let L be a reflexive relational structure. One can verify that $I \longmapsto L$ is reflexive-yielding.

Let I be a non empty set and let L be a non empty antisymmetric relational structure. Note that $\prod(I \longmapsto L)$ is antisymmetric.

Let I be a non empty set and let L be a non empty transitive relational structure. Note that $\prod(I \longmapsto L)$ is transitive.

We now state two propositions:

(13) Let T be a Scott topological augmentation of $2^1_\subseteq$. Then the topology of $T =$ the topology of the Sierpiński space.

(14) Let I be a non empty set. Then $\{\prod((\text{the support of } I \longmapsto \text{the Sierpiński space}) {+\cdot} (i, \{1\})) : i$ ranges over elements of $I\}$ is a prebasis of $\prod(I \longmapsto \text{the Sierpiński space})$.

Let I be a non empty set and let L be a complete lattice. Note that $\prod(I \longmapsto L)$ is complete and has l.u.b.'s.

Let I be a non empty set and let X be an algebraic lower-bounded lattice. Note that $\prod(I \longmapsto X)$ is algebraic.

We now state several propositions:

(15) Let X be a non empty set. Then there exists a map f from $2^X_\subseteq$ into $\prod(X \longmapsto 2^1_\subseteq)$ such that f is isomorphic and for every subset Y of X holds $f(Y) = \chi_{Y,X}$.

(16) Let I be a non empty set and T be a Scott topological augmentation of $\prod(I \longmapsto 2^1_\subseteq)$. Then the topology of $T =$ the topology of $\prod(I \longmapsto$ the Sierpiński space$)$.

(17) Let T, S be non empty topological spaces. Suppose the carrier of $T =$ the carrier of S and the topology of $T =$ the topology of S and T is injective. Then S is injective.

(18) For every non empty set I holds every Scott topological augmentation of $\prod I \longmapsto 2^1_\subseteq$ is injective.

(19) Let T be a T_0-space. Then there exists a non empty set M and there exists a map f from T into $\prod(M \longmapsto$ the Sierpiński space$)$ such that f° is a homeomorphism.

(20) Let T be a T_0-space. Suppose T is injective. Then there exists a non empty set M such that T is a topological retract of $\prod(M \longmapsto$ the Sierpiński space$)$.

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