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Algebraic and Arithmetic Lattices. Part I[1]

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Summary. We formalize [10, pp. 87–89].

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The articles [15], [7], [18], [13], [9], [19], [17], [5], [6], [14], [16], [1], [2], [11], [20], [3], [8], [4], and [12] provide the notation and terminology for this paper.

1. Preliminaries

The scheme *LambdaCD* deals with a non empty set $\mathcal{A}$, a unary functor $\mathcal{F}$ yielding a set, a unary functor $\mathcal{G}$ yielding a set, and a unary predicate $\mathcal{P}$, and states that:

> There exists a function f such that $\operatorname{dom} f = \mathcal{A}$ and for every element x of $\mathcal{A}$ holds if $\mathcal{P}[x]$, then $f(x) = \mathcal{F}(x)$ and if not $\mathcal{P}[x]$, then $f(x) = \mathcal{G}(x)$

for all values of the parameters.

One can prove the following propositions:

(1) Let L be a non empty reflexive transitive relational structure and x, y be elements of L. If $x \leq y$, then $\text{compactbelow}(x) \subseteq \text{compactbelow}(y)$.

(2) For every non empty reflexive relational structure L and for every element x of L holds $\text{compactbelow}(x)$ is a subset of $\text{CompactSublatt}(L)$.

(3) For every relational structure L and for every relational substructure S of L holds every subset of S is a subset of L.

(4) For every non empty reflexive transitive relational structure L with l.u.b.'s holds the carrier of L is an ideal of L.

(5) Let L_1 be a lower-bounded non empty reflexive antisymmetric relational structure and L_2 be a non empty reflexive antisymmetric relational structure. Suppose the relational structure of $L_1 =$ the relational structure of L_2 and L_1 is up-complete. Then the carrier of $\text{CompactSublatt}(L_1) =$ the carrier of $\text{CompactSublatt}(L_2)$.

2. Algebraic and Arithmetic Lattices

One can prove the following three propositions:

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(6) For every algebraic lower-bounded lattice L holds every continuous subframe of L is algebraic.

(7) Let X, E be sets and L be a continuous subframe of $2^X_\subseteq$. Then $E \in$ the carrier of CompactSublatt(L) if and only if there exists an element F of $2^X_\subseteq$ such that F is finite and $E = \bigcap\{Y; Y$ ranges over elements of L: $F \subseteq Y\}$ and $F \subseteq E$.

(8) For every lower-bounded sup-semilattice L holds $\langle \mathrm{Ids}(L), \subseteq \rangle$ is a continuous subframe of $2^{\text{the carrier of } L}_\subseteq$.

Let L be a non empty reflexive transitive relational structure. Observe that there exists an ideal of L which is principal.

Next we state several propositions:

(9) For every lower-bounded sup-semilattice L and for every non empty directed subset X of $\langle \mathrm{Ids}(L), \subseteq \rangle$ holds $\sup X = \bigcup X$.

(10) For every lower-bounded sup-semilattice S holds $\langle \mathrm{Ids}(S), \subseteq \rangle$ is algebraic.

(11) Let S be a lower-bounded sup-semilattice and x be an element of $\langle \mathrm{Ids}(S), \subseteq \rangle$. Then x is compact if and only if x is a principal ideal of S.

(12) Let S be a lower-bounded sup-semilattice and x be an element of $\langle \mathrm{Ids}(S), \subseteq \rangle$. Then x is compact if and only if there exists an element a of S such that $x = {\downarrow} a$.

(13) Let L be a lower-bounded sup-semilattice and f be a map from L into CompactSublatt($\langle \mathrm{Ids}(L), \subseteq \rangle$). If for every element x of L holds $f(x) = {\downarrow} x$, then f is isomorphic.

(14) For every lower-bounded lattice S holds $\langle \mathrm{Ids}(S), \subseteq \rangle$ is arithmetic.

(15) For every lower-bounded sup-semilattice L holds CompactSublatt(L) is a lower-bounded sup-semilattice.

(16) Let L be an algebraic lower-bounded sup-semilattice and f be a map from L into $\langle \mathrm{Ids}(\mathrm{CompactSublatt}(L)), \subseteq \rangle$. If for every element x of L holds $f(x) = \mathrm{compactbelow}(x)$, then f is isomorphic.

(17) Let L be an algebraic lower-bounded sup-semilattice and x be an element of L. Then compactbelow(x) is a principal ideal of CompactSublatt(L) if and only if x is compact.

3. Maps

We now state three propositions:

(18) Let L_1, L_2 be non empty relational structures, X be a subset of L_1, x be an element of L_1, and f be a map from L_1 into L_2. If f is isomorphic, then $x \leq X$ iff $f(x) \leq f^\circ X$.

(19) Let L_1, L_2 be non empty relational structures, X be a subset of L_1, x be an element of L_1, and f be a map from L_1 into L_2. If f is isomorphic, then $x \geq X$ iff $f(x) \geq f^\circ X$.

(20) Let L_1, L_2 be non empty antisymmetric relational structures and f be a map from L_1 into L_2. If f is isomorphic, then f is infs-preserving and sups-preserving.

Let L_1, L_2 be non empty antisymmetric relational structures. One can check that every map from L_1 into L_2 which is isomorphic is also infs-preserving and sups-preserving.

One can prove the following propositions:

(21) Let L_1, L_2, L_3 be non empty transitive antisymmetric relational structures and f be a map from L_1 into L_2. Suppose f is infs-preserving. Suppose L_2 is a full infs-inheriting relational substructure of L_3 and L_3 is complete. Then there exists a map g from L_1 into L_3 such that $f = g$ and g is infs-preserving.

(22) Let L_1, L_2, L_3 be non empty transitive antisymmetric relational structures and f be a map from L_1 into L_2. Suppose f is monotone and directed-sups-preserving. Suppose L_2 is a full directed-sups-inheriting relational substructure of L_3 and L_3 is complete. Then there exists a map g from L_1 into L_3 such that $f = g$ and g is directed-sups-preserving.

(23) For every lower-bounded sup-semilattice L holds $\langle \mathrm{Ids}(\mathrm{CompactSublatt}(L)), \subseteq \rangle$ is a continuous subframe of $2_{\subseteq}^{\text{the carrier of CompactSublatt}(L)}$.

(24) Let L be an algebraic lower-bounded lattice. Then there exists a map g from L into $2_{\subseteq}^{\text{the carrier of CompactSublatt}(L)}$ such that

(i) g is infs-preserving, directed-sups-preserving, and one-to-one, and

(ii) for every element x of L holds $g(x) = \mathrm{compactbelow}(x)$.

(25) Let I be a non empty set and J be a relational structure yielding nonempty reflexive-yielding many sorted set indexed by I. Suppose that for every element i of I holds $J(i)$ is an algebraic lower-bounded lattice. Then $\prod J$ is an algebraic lower-bounded lattice.

(26) Let L_1, L_2 be non empty relational structures. Suppose the relational structure of $L_1 =$ the relational structure of L_2. Then L_1 and L_2 are isomorphic.

(27) Let L_1, L_2 be up-complete non empty posets and f be a map from L_1 into L_2. Suppose f is isomorphic. Let x, y be elements of L_1. Then $x \ll y$ if and only if $f(x) \ll f(y)$.

(28) Let L_1, L_2 be up-complete non empty posets and f be a map from L_1 into L_2. Suppose f is isomorphic. Let x be an element of L_1. Then x is compact if and only if $f(x)$ is compact.

(29) Let L_1, L_2 be up-complete non empty posets and f be a map from L_1 into L_2. If f is isomorphic, then for every element x of L_1 holds $f^{\circ}\mathrm{compactbelow}(x) = \mathrm{compactbelow}(f(x))$.

(30) For all non empty posets L_1, L_2 such that L_1 and L_2 are isomorphic and L_1 is up-complete holds L_2 is up-complete.

(31) For all non empty posets L_1, L_2 such that L_1 and L_2 are isomorphic and L_1 is complete and satisfies axiom K holds L_2 satisfies axiom K.

(32) Let L_1, L_2 be sup-semilattices. Suppose L_1 and L_2 are isomorphic and L_1 is lower-bounded and algebraic. Then L_2 is algebraic.

(33) For every continuous lower-bounded sup-semilattice L holds $\mathrm{SupMap}(L)$ is infs-preserving and sups-preserving.

(34) Let L be a lower-bounded lattice. Then

(i) if L is algebraic, then there exists a non empty set X and there exists a full relational substructure S of $2_{\subseteq}^{X}$ such that S is infs-inheriting and directed-sups-inheriting and L and S are isomorphic, and

(ii) if there exists a set X and there exists a full relational substructure S of $2_{\subseteq}^{X}$ such that S is infs-inheriting and directed-sups-inheriting and L and S are isomorphic, then L is algebraic.

(35) Let L be a lower-bounded lattice. Then

(i) if L is algebraic, then there exists a non empty set X and there exists a closure map c from $2_{\subseteq}^{X}$ into $2_{\subseteq}^{X}$ such that c is directed-sups-preserving and L and $\mathrm{Im}\, c$ are isomorphic, and

(ii) if there exists a set X and there exists a closure map c from $2_{\subseteq}^{X}$ into $2_{\subseteq}^{X}$ such that c is directed-sups-preserving and L and $\mathrm{Im}\, c$ are isomorphic, then L is algebraic.

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