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Algebra of Vector Functions

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Summary. We develop the algebra of partial vector functions, with domains being algebra of vector functions.

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The articles [9], [13], [1], [10], [3], [7], [12], [14], [2], [5], [11], [8], [4], and [6] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: X, Y are sets, C is a non empty set, c is an element of C, V is a real normed space, f, f_1, f_2, f_3 are partial functions from C to the carrier of V, and r, p are real numbers.

Let us consider C, let us consider V, and let us consider f_1, f_2. The functor $f_1 + f_2$ yields a partial function from C to the carrier of V and is defined as follows:

(Def. 1) $\mathrm{dom}(f_1 + f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}(f_1 + f_2)$ holds $(f_1 + f_2)_c = (f_1)_c + (f_2)_c$.

The functor $f_1 - f_2$ yields a partial function from C to the carrier of V and is defined by:

(Def. 2) $\mathrm{dom}(f_1 - f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}(f_1 - f_2)$ holds $(f_1 - f_2)_c = (f_1)_c - (f_2)_c$.

Let us consider C, let us consider V, let f_1 be a partial function from C to $\mathbb{R}$, and let us consider f_2. The functor $f_1\, f_2$ yields a partial function from C to the carrier of V and is defined as follows:

(Def. 3) $\mathrm{dom}(f_1\, f_2) = \mathrm{dom}\, f_1 \cap \mathrm{dom}\, f_2$ and for every c such that $c \in \mathrm{dom}(f_1\, f_2)$ holds $(f_1\, f_2)_c = f_1(c) \cdot (f_2)_c$.

Let us consider C, let us consider V, and let us consider f, r. The functor $r\, f$ yielding a partial function from C to the carrier of V is defined by:

(Def. 4) $\mathrm{dom}(r\, f) = \mathrm{dom}\, f$ and for every c such that $c \in \mathrm{dom}(r\, f)$ holds $(r\, f)_c = r \cdot f_c$.

Let us consider C, let us consider V, and let us consider f. The functor $\|f\|$ yields a partial function from C to $\mathbb{R}$ and is defined as follows:

(Def. 5) $\mathrm{dom}\|f\| = \mathrm{dom}\, f$ and for every c such that $c \in \mathrm{dom}\|f\|$ holds $\|f\|(c) = \|f_c\|$.

The functor $-f$ yields a partial function from C to the carrier of V and is defined by:

(Def. 6) $\mathrm{dom}(-f) = \mathrm{dom}\, f$ and for every c such that $c \in \mathrm{dom}(-f)$ holds $(-f)_c = -f_c$.

One can prove the following propositions:

(7)[1] For every partial function f_1 from C to $\mathbb{R}$ holds $\operatorname{dom}(f_1\, f_2) \setminus (f_1\, f_2)^{-1}(\{0_V\}) = (\operatorname{dom} f_1 \setminus {f_1}^{-1}(\{0\})) \cap (\operatorname{dom} f_2 \setminus {f_2}^{-1}(\{0_V\}))$.

(8) $\|f\|^{-1}(\{0\}) = f^{-1}(\{0_V\})$ and $(-f)^{-1}(\{0_V\}) = f^{-1}(\{0_V\})$.

(9) If $r \neq 0$, then $(r\, f)^{-1}(\{0_V\}) = f^{-1}(\{0_V\})$.

(10) $f_1 + f_2 = f_2 + f_1$.

(11) $(f_1 + f_2) + f_3 = f_1 + (f_2 + f_3)$.

(12) Let f_1, f_2 be partial functions from C to $\mathbb{R}$ and f_3 be a partial function from C to the carrier of V. Then $(f_1\, f_2)\, f_3 = f_1\, (f_2\, f_3)$.

(13) For all partial functions f_1, f_2 from C to $\mathbb{R}$ holds $(f_1 + f_2)\, f_3 = f_1\, f_3 + f_2\, f_3$.

(14) For every partial function f_3 from C to $\mathbb{R}$ holds $f_3\, (f_1 + f_2) = f_3\, f_1 + f_3\, f_2$.

(15) For every partial function f_1 from C to $\mathbb{R}$ holds $r\, (f_1\, f_2) = (r\, f_1)\, f_2$.

(16) For every partial function f_1 from C to $\mathbb{R}$ holds $r\, (f_1\, f_2) = f_1\, (r\, f_2)$.

(17) For all partial functions f_1, f_2 from C to $\mathbb{R}$ holds $(f_1 - f_2)\, f_3 = f_1\, f_3 - f_2\, f_3$.

(18) For every partial function f_3 from C to $\mathbb{R}$ holds $f_3\, f_1 - f_3\, f_2 = f_3\, (f_1 - f_2)$.

(19) $r\, (f_1 + f_2) = r\, f_1 + r\, f_2$.

(20) $(r \cdot p)\, f = r\, (p\, f)$.

(21) $r\, (f_1 - f_2) = r\, f_1 - r\, f_2$.

(22) $f_1 - f_2 = (-1)\, (f_2 - f_1)$.

(23) $f_1 - (f_2 + f_3) = f_1 - f_2 - f_3$.

(24) $1\, f = f$.

(25) $f_1 - (f_2 - f_3) = (f_1 - f_2) + f_3$.

(26) $f_1 + (f_2 - f_3) = (f_1 + f_2) - f_3$.

(27) For every partial function f_1 from C to $\mathbb{R}$ holds $\|f_1\, f_2\| = |f_1|\, \|f_2\|$.

(28) $\|r\, f\| = |r|\, \|f\|$.

(29) $-f = (-1)\, f$.

(30) $--f = f$.

(31) $f_1 - f_2 = f_1 + -f_2$.

(32) $f_1 - -f_2 = f_1 + f_2$.

(33) $(f_1 + f_2){\restriction}X = f_1{\restriction}X + f_2{\restriction}X$ and $(f_1 + f_2){\restriction}X = f_1{\restriction}X + f_2$ and $(f_1 + f_2){\restriction}X = f_1 + f_2{\restriction}X$.

(34) For every partial function f_1 from C to $\mathbb{R}$ holds $(f_1\, f_2){\restriction}X = (f_1{\restriction}X)\, (f_2{\restriction}X)$ and $(f_1\, f_2){\restriction}X = (f_1{\restriction}X)\, f_2$ and $(f_1\, f_2){\restriction}X = f_1\, (f_2{\restriction}X)$.

(35) $(-f){\restriction}X = -f{\restriction}X$ and $\|f\|{\restriction}X = \|f{\restriction}X\|$.

(36) $(f_1 - f_2){\restriction}X = f_1{\restriction}X - f_2{\restriction}X$ and $(f_1 - f_2){\restriction}X = f_1{\restriction}X - f_2$ and $(f_1 - f_2){\restriction}X = f_1 - f_2{\restriction}X$.

(37) $(r\, f){\restriction}X = r\, (f{\restriction}X)$.

[1] The propositions (1)–(6) have been removed.

(38) f_1 is total and f_2 is total iff $f_1 + f_2$ is total and f_1 is total and f_2 is total iff $f_1 - f_2$ is total.

(39) For every partial function f_1 from C to $\mathbb{R}$ holds f_1 is total and f_2 is total iff $f_1\ f_2$ is total.

(40) f is total iff $r\,f$ is total.

(41) f is total iff $-f$ is total.

(42) f is total iff $\|f\|$ is total.

(43) If f_1 is total and f_2 is total, then $(f_1+f_2)_c = (f_1)_c + (f_2)_c$ and $(f_1-f_2)_c = (f_1)_c - (f_2)_c$.

(44) For every partial function f_1 from C to $\mathbb{R}$ such that f_1 is total and f_2 is total holds $(f_1\ f_2)_c = f_1(c) \cdot (f_2)_c$.

(45) If f is total, then $(r\,f)_c = r \cdot f_c$.

(46) If f is total, then $(-f)_c = -f_c$ and $\|f\|(c) = \|f_c\|$.

Let us consider C, let us consider V, and let us consider f, Y. We say that f is bounded on Y if and only if:

(Def. 7) There exists r such that for every c such that $c \in Y \cap \operatorname{dom} f$ holds $\|f_c\| \leq r$.

Next we state a number of propositions:

(48)[2] If $Y \subseteq X$ and f is bounded on X, then f is bounded on Y.

(49) If X misses $\operatorname{dom} f$, then f is bounded on X.

(50) $0\,f$ is bounded on Y.

(51) If f is bounded on Y, then $r\,f$ is bounded on Y.

(52) If f is bounded on Y, then $\|f\|$ is bounded on Y and $-f$ is bounded on Y.

(53) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(54) For every partial function f_1 from C to $\mathbb{R}$ such that f_1 is bounded on X and f_2 is bounded on Y holds $f_1\ f_2$ is bounded on $X \cap Y$.

(55) If f_1 is bounded on X and f_2 is bounded on Y, then $f_1 - f_2$ is bounded on $X \cap Y$.

(56) If f is bounded on X and bounded on Y, then f is bounded on $X \cup Y$.

(57) If f_1 is a constant on X and f_2 is a constant on Y, then $f_1 + f_2$ is a constant on $X \cap Y$ and $f_1 - f_2$ is a constant on $X \cap Y$.

(58) Let f_1 be a partial function from C to $\mathbb{R}$. Suppose f_1 is a constant on X and f_2 is a constant on Y. Then $f_1\ f_2$ is a constant on $X \cap Y$.

(59) If f is a constant on Y, then $p\,f$ is a constant on Y.

(60) If f is a constant on Y, then $\|f\|$ is a constant on Y and $-f$ is a constant on Y.

(61) If f is a constant on Y, then f is bounded on Y.

(62) If f is a constant on Y, then for every r holds $r\,f$ is bounded on Y and $-f$ is bounded on Y and $\|f\|$ is bounded on Y.

(63) If f_1 is bounded on X and f_2 is a constant on Y, then $f_1 + f_2$ is bounded on $X \cap Y$.

(64) If f_1 is bounded on X and f_2 is a constant on Y, then $f_1 - f_2$ is bounded on $X \cap Y$ and $f_2 - f_1$ is bounded on $X \cap Y$.

[2] The proposition (47) has been removed.

REFERENCES

[1] Grzegorz Bancerek. The ordinal numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/ordinal1.html`.

[2] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/partfun1.html`.

[3] Krzysztof Hryniewiecki. Basic properties of real numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/real_1.html`.

[4] Jarosław Kotowicz. Real sequences and basic operations on them. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/seq_1.html`.

[5] Jarosław Kotowicz. Partial functions from a domain to a domain. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/partfun2.html`.

[6] Jarosław Kotowicz. Partial functions from a domain to the set of real numbers. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/rfunct_1.html`.

[7] Jan Popiołek. Some properties of functions modul and signum. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/absvalue.html`.

[8] Jan Popiołek. Real normed space. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/normsp_1.html`.

[9] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[10] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[11] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/rlvect_1.html`.

[12] Wojciech A. Trybulec. Pigeon hole principle. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseq_4.html`.

[13] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[14] Edmund Woronowicz. Relations defined on sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relset_1.html`.

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