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Subspaces and Cosets of Subspaces in Vector Space

Wojciech A. Trybulec
Warsaw University

Summary. We introduce the notions of subspace of vector space and coset of a subspace. We prove a number of theorems concerning those notions. Some theorems that belong rather to [4] are proved.

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The articles [6], [3], [8], [9], [1], [2], [5], [7], and [4] provide the notation and terminology for this paper.

In this paper x is a set.

Let G_1 be a non empty groupoid, let V be a non empty vector space structure over G_1, and let V_1 be a subset of V. We say that V_1 is linearly closed if and only if the conditions (Def. 1) are satisfied.

(Def. 1)(i) For all elements v, u of V such that $v \in V_1$ and $u \in V_1$ holds $v + u \in V_1$, and

(ii) for every element a of G_1 and for every element v of V such that $v \in V_1$ holds $a \cdot v \in V_1$.

One can prove the following propositions:

(4)[1] Let G_1 be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be an Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1, and V_1 be a subset of V. If $V_1 \neq \emptyset$ and V_1 is linearly closed, then $0_V \in V_1$.

(5) Let G_1 be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be an Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1, and V_1 be a subset of V. Suppose V_1 is linearly closed. Let v be an element of V. If $v \in V_1$, then $-v \in V_1$.

(6) Let G_1 be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be an Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1, and V_1 be a subset of V. Suppose V_1 is linearly closed. Let v, u be elements of V. If $v \in V_1$ and $u \in V_1$, then $v - u \in V_1$.

(7) Let G_1 be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure and V be an Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1. Then $\{0_V\}$ is linearly closed.

[1] The propositions (1)–(3) have been removed.

(8) Let G_1 be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be an Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1, and V_1 be a subset of V. If the carrier of $V = V_1$, then V_1 is linearly closed.

(9) Let G_1 be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be an Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1, and V_1, V_2, V_3 be subsets of V. Suppose V_1 is linearly closed and V_2 is linearly closed and $V_3 = \{v+u; v$ ranges over elements of V, u ranges over elements of $V\colon\ v \in V_1 \ \wedge\ u \in V_2\}$. Then V_3 is linearly closed.

(10) Let G_1 be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be an Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1, and V_1, V_2 be subsets of V. Suppose V_1 is linearly closed and V_2 is linearly closed. Then $V_1 \cap V_2$ is linearly closed.

Let G_1 be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure and let V be an Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1. An Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1 is said to be a subspace of V if it satisfies the conditions (Def. 2).

(Def. 2)(i) The carrier of it $\subseteq$ the carrier of V,

(ii) the zero of it $=$ the zero of V,

(iii) the addition of it $=$ (the addition of V)$\restriction$[: the carrier of it, the carrier of it :], and

(iv) the left multiplication of it $=$ (the left multiplication of V)$\restriction$[: the carrier of G_1, the carrier of it :].

For simplicity, we use the following convention: G_1 is an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V, X, Y are Abelian add-associative right zeroed right complementable vector space-like non empty vector space structures over G_1, a is an element of G_1, u, v, v_1, v_2 are elements of V, W, W_1, W_2 are subspaces of V, V_1 is a subset of V, and w, w_1, w_2 are elements of W.

We now state a number of propositions:

(16)[2] If $x \in W_1$ and W_1 is a subspace of W_2, then $x \in W_2$.

(17) If $x \in W$, then $x \in V$.

(18) w is an element of V.

(19) $0_W = 0_V$.

(20) $0_{(W_1)} = 0_{(W_2)}$.

(21) If $w_1 = v$ and $w_2 = u$, then $w_1 + w_2 = v + u$.

(22) If $w = v$, then $a \cdot w = a \cdot v$.

(23) If $w = v$, then $-v = -w$.

(24) If $w_1 = v$ and $w_2 = u$, then $w_1 - w_2 = v - u$.

(25) $0_V \in W$.

(26) $0_{(W_1)} \in W_2$.

[2] The propositions (11)–(15) have been removed.

(27) $0_W \in V$.

(28) If $u \in W$ and $v \in W$, then $u + v \in W$.

(29) If $v \in W$, then $a \cdot v \in W$.

(30) If $v \in W$, then $-v \in W$.

(31) If $u \in W$ and $v \in W$, then $u - v \in W$.

(32) V is a subspace of V.

(33) Let X, V be strict Abelian add-associative right zeroed right complementable vector space-like non empty vector space structures over G_1. If V is a subspace of X and X is a subspace of V, then $V = X$.

(34) If V is a subspace of X and X is a subspace of Y, then V is a subspace of Y.

(35) If the carrier of $W_1 \subseteq$ the carrier of W_2, then W_1 is a subspace of W_2.

(36) If for every v such that $v \in W_1$ holds $v \in W_2$, then W_1 is a subspace of W_2.

Let G_1 be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure and let V be an Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1. Note that there exists a subspace of V which is strict.

Next we state several propositions:

(37) For all strict subspaces W_1, W_2 of V such that the carrier of $W_1 =$ the carrier of W_2 holds $W_1 = W_2$.

(38) For all strict subspaces W_1, W_2 of V such that for every v holds $v \in W_1$ iff $v \in W_2$ holds $W_1 = W_2$.

(39) Let V be a strict Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1 and W be a strict subspace of V. If the carrier of $W =$ the carrier of V, then $W = V$.

(40) Let V be a strict Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1 and W be a strict subspace of V. If for every element v of V holds $v \in W$, then $W = V$.

(41) If the carrier of $W = V_1$, then V_1 is linearly closed.

(42) If $V_1 \neq \emptyset$ and V_1 is linearly closed, then there exists a strict subspace W of V such that $V_1 =$ the carrier of W.

Let us consider G_1 and let us consider V. The functor $\mathbf{0}_V$ yields a strict subspace of V and is defined by:

(Def. 3) The carrier of $\mathbf{0}_V = \{0_V\}$.

Let us consider G_1 and let us consider V. The functor Ω_V yielding a strict subspace of V is defined as follows:

(Def. 4) $\Omega_V =$ the vector space structure of V.

The following propositions are true:

(46)[3] $x \in \mathbf{0}_V$ iff $x = 0_V$.

(47) $\mathbf{0}_W = \mathbf{0}_V$.

[3] The propositions (43)–(45) have been removed.

(48) $\mathbf{0}_{(W_1)} = \mathbf{0}_{(W_2)}$.

(49) $\mathbf{0}_W$ is a subspace of V.

(50) $\mathbf{0}_V$ is a subspace of W.

(51) $\mathbf{0}_{(W_1)}$ is a subspace of W_2.

(53)[4] Every strict Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure V over G_1 is a subspace of Ω_V.

Let us consider G_1, let us consider V, and let us consider v, W. The functor $v+W$ yielding a subset of V is defined by:

(Def. 5) $v+W = \{v+u : u \in W\}$.

Let us consider G_1, let us consider V, and let us consider W. A subset of V is called a coset of W if:

(Def. 6) There exists v such that it $= v+W$.

In the sequel B, C denote cosets of W.
We now state a number of propositions:

(57)[5] $x \in v+W$ iff there exists u such that $u \in W$ and $x = v+u$.

(58) $0_V \in v+W$ iff $v \in W$.

(59) $v \in v+W$.

(60) $0_V + W =$ the carrier of W.

(61) $v + \mathbf{0}_V = \{v\}$.

(62) $v + \Omega_V =$ the carrier of V.

(63) $0_V \in v+W$ iff $v+W =$ the carrier of W.

(64) $v \in W$ iff $v+W =$ the carrier of W.

(65) If $v \in W$, then $a \cdot v + W =$ the carrier of W.

(66) Let G_1 be a field, V be a vector space over G_1, a be an element of G_1, v be an element of V, and W be a subspace of V. If $a \neq 0_{(G_1)}$ and $a \cdot v + W =$ the carrier of W, then $v \in W$.

(67) Let G_1 be a field, V be a vector space over G_1, v be an element of V, and W be a subspace of V. Then $v \in W$ if and only if $-v+W =$ the carrier of W.

(68) $u \in W$ iff $v+W = v+u+W$.

(69) $u \in W$ iff $v+W = (v-u)+W$.

(70) $v \in u+W$ iff $u+W = v+W$.

(71) If $u \in v_1+W$ and $u \in v_2+W$, then $v_1+W = v_2+W$.

(72) Let G_1 be a field, V be a vector space over G_1, a be an element of G_1, v be an element of V, and W be a subspace of V. If $a \neq \mathbf{1}_{(G_1)}$ and $a \cdot v \in v+W$, then $v \in W$.

(73) If $v \in W$, then $a \cdot v \in v+W$.

(74) If $v \in W$, then $-v \in v+W$.

[4] The proposition (52) has been removed.
[5] The propositions (54)–(56) have been removed.

(75) $u+v \in v+W$ iff $u \in W$.

(76) $v-u \in v+W$ iff $u \in W$.

(78)[6] $u \in v+W$ iff there exists v_1 such that $v_1 \in W$ and $u = v - v_1$.

(79) There exists v such that $v_1 \in v+W$ and $v_2 \in v+W$ iff $v_1 - v_2 \in W$.

(80) If $v+W = u+W$, then there exists v_1 such that $v_1 \in W$ and $v+v_1 = u$.

(81) If $v+W = u+W$, then there exists v_1 such that $v_1 \in W$ and $v-v_1 = u$.

(82) For all strict subspaces W_1, W_2 of V holds $v+W_1 = v+W_2$ iff $W_1 = W_2$.

(83) For all strict subspaces W_1, W_2 of V such that $v+W_1 = u+W_2$ holds $W_1 = W_2$.

(84) There exists C such that $v \in C$.

(85) C is linearly closed iff $C =$ the carrier of W.

(86) For all strict subspaces W_1, W_2 of V and for every coset C_1 of W_1 and for every coset C_2 of W_2 such that $C_1 = C_2$ holds $W_1 = W_2$.

(87) $\{v\}$ is a coset of $\mathbf{0}_V$.

(88) If V_1 is a coset of $\mathbf{0}_V$, then there exists v such that $V_1 = \{v\}$.

(89) The carrier of W is a coset of W.

(90) The carrier of V is a coset of Ω_V.

(91) If V_1 is a coset of Ω_V, then $V_1 =$ the carrier of V.

(92) $0_V \in C$ iff $C =$ the carrier of W.

(93) $u \in C$ iff $C = u+W$.

(94) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u+v_1 = v$.

(95) If $u \in C$ and $v \in C$, then there exists v_1 such that $v_1 \in W$ and $u-v_1 = v$.

(96) There exists C such that $v_1 \in C$ and $v_2 \in C$ iff $v_1 - v_2 \in W$.

(97) If $u \in B$ and $u \in C$, then $B = C$.

(103)[7] Let G_1 be an add-associative right zeroed right complementable Abelian commutative associative left unital distributive non empty double loop structure, V be an Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over G_1, a, b be elements of G_1, and v be an element of V. Then $(a-b) \cdot v = a \cdot v - b \cdot v$.

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[6] The proposition (77) has been removed.

[7] The propositions (98)–(102) have been removed.

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