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# Construction of Rings and Left-, Right-, and Bi-Modules over a Ring

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**Summary.** Definitions of some classes of rings and left-, right-, and bi-modules over a ring and some elementary theorems on rings and skew fields.

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The articles [7], [4], [9], [3], [2], [8], [5], [6], and [1] provide the notation and terminology for this paper.

In this paper $F_1$ denotes a non empty double loop structure and $F$ denotes a field.

Let $I_1$ be a non empty multiplicative loop structure. We say that $I_1$ is well unital if and only if:

(Def. 2)[1] For every element $x$ of $I_1$ holds $x \cdot \mathbf{1}_{(I_1)} = x$ and $\mathbf{1}_{(I_1)} \cdot x = x$.

One can verify the following observations:

* every non empty multiplicative loop structure which is well unital is also left unital and right unital,
* every non empty multiplicative loop structure which is left unital and right unital is also well unital, and
* there exists a non empty double loop structure which is strict, Abelian, add-associative, right zeroed, right complementable, well unital, and distributive.

The following proposition is true

(1) For all scalars $x$, $y$, $z$ of $F_1$ holds $x+y=y+x$ and $(x+y)+z=x+(y+z)$ and $x+0_{(F_1)}=x$ and $x+-x=0_{(F_1)}$ and $x\cdot\mathbf{1}_{(F_1)}=x$ and $\mathbf{1}_{(F_1)}\cdot x=x$ and $(x\cdot y)\cdot z=x\cdot(y\cdot z)$ and $x\cdot(y+z)=x\cdot y+x\cdot z$ and $(y+z)\cdot x=y\cdot x+z\cdot x$ if and only if $F_1$ is a ring.

Let us note that there exists a ring which is strict.

Let us note that there exists a ring which is commutative.

A commutative ring is a commutative ring.

One can check that there exists a commutative ring which is strict.

Let $I_1$ be a non empty multiplicative loop with zero structure. We say that $I_1$ is integral domain-like if and only if:

(Def. 5)[2] For all elements $x$, $y$ of $I_1$ such that $x\cdot y=0_{(I_1)}$ holds $x=0_{(I_1)}$ or $y=0_{(I_1)}$.

[1] The definition (Def. 1) has been removed.
[2] The definitions (Def. 3) and (Def. 4) have been removed.

One can check that there exists a commutative ring which is strict, non degenerated, and integral domain-like.

An integral domain is an integral domain-like non degenerated commutative ring.

Next we state the proposition

(13)[3] $F$ is an integral domain.

Let us mention that there exists a ring which is non degenerated and field-like.

A skew field is a non degenerated field-like ring.

Let us mention that there exists a skew field which is strict.

In the sequel $R$ is a ring.

The following propositions are true:

(16)[4] Suppose that for every scalar $x$ of $R$ holds if $x \neq 0_R$, then there exists a scalar $y$ of $R$ such that $x \cdot y = \mathbf{1}_R$ and $0_R \neq \mathbf{1}_R$. Then $R$ is a skew field.

(19)[5] $F$ is a skew field.

Let us note that every non empty multiplicative loop structure which is commutative and left unital is also well unital and every non empty multiplicative loop structure which is commutative and right unital is also well unital.

In the sequel $R$ denotes an Abelian add-associative right zeroed right complementable non empty loop structure and $x$, $y$, $z$ denote scalars of $R$.

The following three propositions are true:

(22)[6] $x+y=z$ iff $x=z-y$ and $x+y=z$ iff $y=z-x$.

(34)[7] Let $R$ be an add-associative right zeroed right complementable non empty loop structure and $x$ be an element of $R$. Then $x=0_R$ if and only if $-x=0_R$.

(38)[8] Let $R$ be an add-associative right zeroed Abelian right complementable non empty loop structure and $x$, $y$ be elements of $R$. Then there exists an element $z$ of $R$ such that $x=y+z$ and $x=z+y$.

In the sequel $S_1$ denotes a skew field and $x$, $y$, $z$ denote scalars of $S_1$.

The following propositions are true:

(39) Let $F$ be an add-associative right zeroed right complementable distributive non degenerated non empty double loop structure and $x$, $y$ be elements of $F$. If $x \cdot y = \mathbf{1}_F$, then $x \neq 0_F$ and $y \neq 0_F$.

(40) Let $S_1$ be a non degenerated field-like associative Abelian add-associative right zeroed right complementable well unital distributive non empty double loop structure and $x$ be an element of $S_1$. If $x \neq 0_{(S_1)}$, then there exists an element $y$ of $S_1$ such that $y \cdot x = \mathbf{1}_{(S_1)}$.

(41) If $x \cdot y = \mathbf{1}_{(S_1)}$, then $y \cdot x = \mathbf{1}_{(S_1)}$.

(42) Let $S_1$ be a non degenerated field-like associative Abelian add-associative right zeroed right complementable well unital distributive non empty double loop structure and $x$, $y$, $z$ be elements of $S_1$. If $x \cdot y = x \cdot z$ and $x \neq 0_{(S_1)}$, then $y=z$.

Let $S_1$ be a non degenerated field-like associative Abelian add-associative right zeroed right complementable well unital distributive non empty double loop structure and let $x$ be an element of $S_1$. Let us assume that $x \neq 0_{(S_1)}$. The functor $x^{-1}$ yielding a scalar of $S_1$ is defined as follows:

[3] The propositions (2)–(12) have been removed.
[4] The propositions (14) and (15) have been removed.
[5] The propositions (17) and (18) have been removed.
[6] The propositions (20) and (21) have been removed.
[7] The propositions (23)–(33) have been removed.
[8] The propositions (35)–(37) have been removed.

(Def. 7)[9] $x \cdot x^{-1} = \mathbf{1}_{(S_1)}$.

Let us consider $S_1$, $x$, $y$. The functor $\frac{x}{y}$ yielding a scalar of $S_1$ is defined by:

(Def. 8) $\frac{x}{y} = x \cdot y^{-1}$.

One can prove the following propositions:

(43) If $x \neq 0_{(S_1)}$, then $x \cdot x^{-1} = \mathbf{1}_{(S_1)}$ and $x^{-1} \cdot x = \mathbf{1}_{(S_1)}$.

(45)[10] If $x \cdot y = \mathbf{1}_{(S_1)}$, then $x = y^{-1}$ and $y = x^{-1}$.

(46) If $x \neq 0_{(S_1)}$ and $y \neq 0_{(S_1)}$, then $x^{-1} \cdot y^{-1} = (y \cdot x)^{-1}$.

(47) If $x \cdot y = 0_{(S_1)}$, then $x = 0_{(S_1)}$ or $y = 0_{(S_1)}$.

(48) If $x \neq 0_{(S_1)}$, then $x^{-1} \neq 0_{(S_1)}$.

(49) If $x \neq 0_{(S_1)}$, then $(x^{-1})^{-1} = x$.

(50) If $x \neq 0_{(S_1)}$, then $\frac{\mathbf{1}_{(S_1)}}{x} = x^{-1}$ and $\frac{\mathbf{1}_{(S_1)}}{x^{-1}} = x$.

(51) If $x \neq 0_{(S_1)}$, then $x \cdot \frac{\mathbf{1}_{(S_1)}}{x} = \mathbf{1}_{(S_1)}$ and $\frac{\mathbf{1}_{(S_1)}}{x} \cdot x = \mathbf{1}_{(S_1)}$.

(52) If $x \neq 0_{(S_1)}$, then $\frac{x}{x} = \mathbf{1}_{(S_1)}$.

(53) If $y \neq 0_{(S_1)}$ and $z \neq 0_{(S_1)}$, then $\frac{x}{y} = \frac{x \cdot z}{y \cdot z}$.

(54) If $y \neq 0_{(S_1)}$, then $-\frac{x}{y} = \frac{-x}{y}$ and $\frac{x}{-y} = -\frac{x}{y}$.

(55) If $z \neq 0_{(S_1)}$, then $\frac{x}{z} + \frac{y}{z} = \frac{x+y}{z}$ and $\frac{x}{z} - \frac{y}{z} = \frac{x-y}{z}$.

(56) If $y \neq 0_{(S_1)}$ and $z \neq 0_{(S_1)}$, then $\frac{x}{\frac{y}{z}} = \frac{x \cdot z}{y}$.

(57) If $y \neq 0_{(S_1)}$, then $\frac{x}{y} \cdot y = x$.

Let $F_1$ be a 1-sorted structure. We consider right module structures over $F_1$ as extensions of loop structure as systems

⟨ a carrier, an addition, a zero, a right multiplication ⟩,

where the carrier is a set, the addition is a binary operation on the carrier, the zero is an element of the carrier, and the right multiplication is a function from [: the carrier, the carrier of $F_1$ :] into the carrier.

Let $F_1$ be a 1-sorted structure. Observe that there exists a right module structure over $F_1$ which is non empty.

Let $F_1$ be a 1-sorted structure, let $A$ be a non empty set, let $a$ be a binary operation on $A$, let $Z$ be an element of $A$, and let $r$ be a function from [: $A$, the carrier of $F_1$ :] into $A$. Note that $\langle A, a, Z, r \rangle$ is non empty.

Let us consider $F_1$ and let $R_1$ be a non empty right module structure over $F_1$. A scalar of $R_1$ is an element of $F_1$. A vector of $R_1$ is an element of $R_1$.

Let $F_2$, $F_3$ be 1-sorted structures. We introduce bimodule structures over $F_2$, $F_3$ which are extensions of vector space structure over $F_2$ and right module structure over $F_3$ and are systems

⟨ a carrier, an addition, a zero, a left multiplication, a right multiplication ⟩,

where the carrier is a set, the addition is a binary operation on the carrier, the zero is an element of the carrier, the left multiplication is a function from [: the carrier of $F_2$, the carrier :] into the carrier, and the right multiplication is a function from [: the carrier, the carrier of $F_3$ :] into the carrier.

Let $F_2$, $F_3$ be 1-sorted structures. Observe that there exists a bimodule structure over $F_2$, $F_3$ which is non empty.

[9] The definition (Def. 6) has been removed.

[10] The proposition (44) has been removed.

Let $F_2$, $F_3$ be 1-sorted structures, let $A$ be a non empty set, let $a$ be a binary operation on $A$, let $Z$ be an element of $A$, let $l$ be a function from $[:$the carrier of $F_2$, $A:]$ into $A$, and let $r$ be a function from $[:A$, the carrier of $F_3:]$ into $A$. Note that $\langle A, a, Z, l, r \rangle$ is non empty.

In the sequel $R$, $R_2$, $R_3$ are rings.

Let $R$ be an Abelian add-associative right zeroed right complementable non empty loop structure. The functor AbGr($R$) yielding a strict Abelian group is defined as follows:

(Def. 9) AbGr($R$) = $\langle$the carrier of $R$, the addition of $R$, the zero of $R\rangle$.

Let us consider $R$. Note that there exists a non empty vector space structure over $R$ which is Abelian, add-associative, right zeroed, right complementable, and strict.

Let us consider $R$. The functor LeftMod($R$) yielding an Abelian add-associative right zeroed right complementable strict non empty vector space structure over $R$ is defined as follows:

(Def. 11)[11] LeftMod($R$) = $\langle$the carrier of $R$, the addition of $R$, the zero of $R$, the multiplication of $R\rangle$.

Let us consider $R$. Note that there exists a non empty right module structure over $R$ which is Abelian, add-associative, right zeroed, right complementable, and strict.

Let us consider $R$. The functor RightMod($R$) yields an Abelian add-associative right zeroed right complementable strict non empty right module structure over $R$ and is defined by:

(Def. 14)[12] RightMod($R$) = $\langle$the carrier of $R$, the addition of $R$, the zero of $R$, the multiplication of $R\rangle$.

Let $R$ be a non empty 1-sorted structure, let $V$ be a non empty right module structure over $R$, let $x$ be an element of $R$, and let $v$ be an element of $V$. The functor $v \cdot x$ yields an element of $V$ and is defined as follows:

(Def. 15) $v \cdot x =$ (the right multiplication of $V$)$(v, x)$.

(Def. 17) $\mathrm{op}_1$ is a unary operation on $\{\emptyset\}$.

(Def. 18) $\mathrm{op}_0$ is an element of $\{\emptyset\}$.

Let us consider $R_2$, $R_3$. Note that there exists a non empty bimodule structure over $R_2$, $R_3$ which is Abelian, add-associative, right zeroed, right complementable, and strict.

Let us consider $R_2$, $R_3$. The functor BiMod($R_2$,$R_3$) yields an Abelian add-associative right zeroed right complementable strict non empty bimodule structure over $R_2$, $R_3$ and is defined by:

(Def. 21)[13] $\mathrm{BiMod}(R_2,R_3) = \langle \{\emptyset\}, \mathrm{op}_2, \mathrm{op}_0, \pi_2((\text{the carrier of } R_2) \times \{\emptyset\}), \pi_1(\{\emptyset\} \times \text{the carrier of } R_3)\rangle$.

One can prove the following proposition

(71)[14] Let $x$, $y$ be scalars of $R$ and $v$, $w$ be vectors of LeftMod($R$). Then $x \cdot (v + w) = x \cdot v + x \cdot w$ and $(x + y) \cdot v = x \cdot v + y \cdot v$ and $(x \cdot y) \cdot v = x \cdot (y \cdot v)$ and $\mathbf{1}_R \cdot v = v$.

Let us consider $R$. One can check that there exists a non empty vector space structure over $R$ which is vector space-like, Abelian, add-associative, right zeroed, right complementable, and strict.

Let us consider $R$. A left module over $R$ is an Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over $R$.

Let us consider $R$. Observe that LeftMod($R$) is Abelian, add-associative, right zeroed, right complementable, strict, and vector space-like.

One can prove the following proposition

(77)[15] Let $x$, $y$ be scalars of $R$ and $v$, $w$ be vectors of RightMod($R$). Then $(v + w) \cdot x = v \cdot x + w \cdot x$ and $v \cdot (x + y) = v \cdot x + v \cdot y$ and $v \cdot (y \cdot x) = (v \cdot y) \cdot x$ and $v \cdot \mathbf{1}_R = v$.

[11] The definition (Def. 10) has been removed.
[12] The definitions (Def. 12) and (Def. 13) have been removed.
[13] The definitions (Def. 18)–(Def. 20) have been removed.
[14] The propositions (58)–(70) have been removed.
[15] The propositions (72)–(76) have been removed.

Let $R$ be a non empty double loop structure and let $I_1$ be a non empty right module structure over $R$. We say that $I_1$ is right module-like if and only if the condition (Def. 23) is satisfied.

(Def. 23)[16] Let $x$, $y$ be scalars of $R$ and $v$, $w$ be vectors of $I_1$. Then $(v+w)\cdot x = v\cdot x + w\cdot x$ and $v\cdot(x+y) = v\cdot x + v\cdot y$ and $v\cdot(y\cdot x) = (v\cdot y)\cdot x$ and $v\cdot \mathbf{1}_R = v$.

Let us consider $R$. Observe that there exists a non empty right module structure over $R$ which is Abelian, add-associative, right zeroed, right complementable, right module-like, and strict.

Let us consider $R$. A right module over $R$ is an Abelian add-associative right zeroed right complementable right module-like non empty right module structure over $R$.

Let us consider $R$. One can verify that $\mathrm{RightMod}(R)$ is Abelian, add-associative, right zeroed, right complementable, and right module-like.

Let us consider $R_2$, $R_3$ and let $I_1$ be a non empty bimodule structure over $R_2$, $R_3$. We say that $I_1$ is bimodule-like if and only if:

(Def. 24) For every scalar $x$ of $R_2$ and for every scalar $p$ of $R_3$ and for every vector $v$ of $I_1$ holds $x\cdot(v\cdot p) = (x\cdot v)\cdot p$.

Let us consider $R_2$, $R_3$. Note that there exists a non empty bimodule structure over $R_2$, $R_3$ which is Abelian, add-associative, right zeroed, right complementable, right module-like, vector space-like, bimodule-like, and strict.

Let us consider $R_2$, $R_3$. A bimodule over $R_2$ and $R_3$ is an Abelian add-associative right zeroed right complementable right module-like vector space-like bimodule-like non empty bimodule structure over $R_2$, $R_3$.

Next we state two propositions:

(83)[17] Let $V$ be a non empty bimodule structure over $R_2$, $R_3$. Then the following statements are equivalent

(i) for all scalars $x$, $y$ of $R_2$ and for all scalars $p$, $q$ of $R_3$ and for all vectors $v$, $w$ of $V$ holds $x\cdot(v+w) = x\cdot v + x\cdot w$ and $(x+y)\cdot v = x\cdot v + y\cdot v$ and $(x\cdot y)\cdot v = x\cdot(y\cdot v)$ and $\mathbf{1}_{(R_2)}\cdot v = v$ and $(v+w)\cdot p = v\cdot p + w\cdot p$ and $v\cdot(p+q) = v\cdot p + v\cdot q$ and $v\cdot(q\cdot p) = (v\cdot q)\cdot p$ and $v\cdot \mathbf{1}_{(R_3)} = v$ and $x\cdot(v\cdot p) = (x\cdot v)\cdot p$,

(ii) $V$ is right module-like, vector space-like, and bimodule-like.

(84) $\mathrm{BiMod}(R_2,R_3)$ is a bimodule over $R_2$ and $R_3$.

Let us consider $R_2$, $R_3$. One can verify that $\mathrm{BiMod}(R_2,R_3)$ is Abelian, add-associative, right zeroed, right complementable, right module-like, vector space-like, and bimodule-like.

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[17] The propositions (78)–(82) have been removed.

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