JOURNAL OF FORMALIZED MATHEMATICS
Volume 14, Released 2002, Published 2003
Inst. of Computer Science, Univ. of Białystok

Quotient Vector Spaces and Functionals[1]

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Summary. The article presents well known facts about quotient vector spaces and functionals (see [8]). There are repeated theorems and constructions with either weaker assumptions or in more general situations (see [11], [7], [10]). The construction of coefficient functionals and non degenerated functional in quotient vector space generated by functional in the given vector space are the only new things which are done.

MML Identifier: `VECTSP10`.

WWW: `http://mizar.org/JFM/Vol14/vectsp10.html`

The articles [13], [5], [20], [12], [3], [1], [15], [2], [17], [7], [21], [4], [6], [14], [19], [18], [16], and [9] provide the notation and terminology for this paper.

1. Auxiliary Facts about Double Loops and Vector Spaces

The following proposition is true

(1) Let K be an add-associative right zeroed right complementable left distributive left unital non empty double loop structure and a be an element of K. Then $(-\mathbf{1}_K) \cdot a = -a$.

Let K be a double loop structure. The functor StructVectSp(K) yields a strict vector space structure over K and is defined as follows:

(Def. 1) StructVectSp(K) $= \langle$the carrier of K, the addition of K, the zero of K, the multiplication of $K\rangle$.

Let K be a non empty double loop structure. Observe that StructVectSp(K) is non empty.

Let K be an Abelian non empty double loop structure. Observe that StructVectSp(K) is Abelian.

Let K be an add-associative non empty double loop structure. Note that StructVectSp(K) is add-associative.

Let K be a right zeroed non empty double loop structure. Note that StructVectSp(K) is right zeroed.

Let K be a right complementable non empty double loop structure. Note that StructVectSp(K) is right complementable.

Let K be an associative left unital distributive non empty double loop structure. Observe that StructVectSp(K) is vector space-like.

Let K be a non degenerated non empty double loop structure. Observe that StructVectSp(K) is non trivial.

Let K be a non degenerated non empty double loop structure. One can check that there exists a non empty vector space structure over K which is non trivial.

[1]This work has been partially supported by TRIAL-SOLUTION grant IST-2001-35447 and SPUB-M grant of KBN.

Let K be an add-associative right zeroed right complementable non empty double loop structure. One can verify that there exists a non empty vector space structure over K which is add-associative, right zeroed, right complementable, and strict.

Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure. One can check that there exists a non empty vector space structure over K which is add-associative, right zeroed, right complementable, vector space-like, and strict.

Let K be an Abelian add-associative right zeroed right complementable associative left unital distributive non degenerated non empty double loop structure. Observe that there exists a non empty vector space structure over K which is Abelian, add-associative, right zeroed, right complementable, vector space-like, strict, and non trivial.

Next we state a number of propositions:

(2) Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, a be an element of K, V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K, and v be a vector of V. Then $0_K \cdot v = 0_V$ and $a \cdot 0_V = 0_V$.

(3) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, S, T be subspaces of V, and v be a vector of V. If $S \cap T = \mathbf{0}_V$ and $v \in S$ and $v \in T$, then $v = 0_V$.

(4) Let K be a field, V be a vector space over K, x be a set, and v be a vector of V. Then $x \in \mathrm{Lin}(\{v\})$ if and only if there exists an element a of K such that $x = a \cdot v$.

(5) Let K be a field, V be a vector space over K, v be a vector of V, and a, b be scalars of V. If $v \neq 0_V$ and $a \cdot v = b \cdot v$, then $a = b$.

(6) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_1 \in W_1$ and $v_2 \in W_2$ and $v = v_1 + v_2$, then $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$.

(7) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$, then $v = v_1 + v_2$.

(8) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$, then $v_1 \in W_1$ and $v_2 \in W_2$.

(9) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v, v_1, v_2 be vectors of V. If $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$, then $v_{\langle W_2, W_1 \rangle} = \langle v_2, v_1 \rangle$.

(10) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v be a vector of V. If $v \in W_1$, then $v_{\langle W_1, W_2 \rangle} = \langle v, 0_V \rangle$.

(11) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, and W_1, W_2 be subspaces of V. Suppose V is the direct sum of W_1 and W_2. Let v be a vector of V. If $v \in W_2$, then $v_{\langle W_1, W_2 \rangle} = \langle 0_V, v \rangle$.

(12) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, V_1 be a subspace of V, W_1 be a subspace of V_1, and v be a vector of V. If $v \in W_1$, then v is a vector of V_1.

(13) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, V_1, V_2, W be subspaces of V, and W_1, W_2 be subspaces of W. If $W_1 = V_1$ and $W_2 = V_2$, then $W_1 + W_2 = V_1 + V_2$.

(14) Let K be a field, V be a vector space over K, W be a subspace of V, v be a vector of V, and w be a vector of W. If $v = w$, then $\mathrm{Lin}(\{w\}) = \mathrm{Lin}(\{v\})$.

(15) Let K be a field, V be a vector space over K, v be a vector of V, and X be a subspace of V. Suppose $v \notin X$. Let y be a vector of $X + \mathrm{Lin}(\{v\})$ and W be a subspace of $X + \mathrm{Lin}(\{v\})$. If $v = y$ and $W = X$, then $X + \mathrm{Lin}(\{v\})$ is the direct sum of W and $\mathrm{Lin}(\{y\})$.

(16) Let K be a field, V be a vector space over K, v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. If $v = y$ and $X = W$ and $v \notin X$, then $y_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle 0_W, y \rangle$.

(17) Let K be a field, V be a vector space over K, v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$. If $w \in X$, then $w_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle w, 0_V \rangle$.

(18) Let K be an add-associative right zeroed right complementable associative Abelian left unital distributive non empty double loop structure, V be a vector space over K, v be a vector of V, and W_1, W_2 be subspaces of V. Then there exist vectors v_1, v_2 of V such that $v_{\langle W_1, W_2 \rangle} = \langle v_1, v_2 \rangle$.

(19) Let K be a field, V be a vector space over K, v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$. Then there exists a vector x of X and there exists an element r of K such that $w_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle x, r \cdot v \rangle$.

(20) Let K be a field, V be a vector space over K, v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w_1, w_2 be vectors of $X + \mathrm{Lin}(\{v\})$, x_1, x_2 be vectors of X, and r_1, r_2 be elements of K. If $(w_1)_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle x_1, r_1 \cdot v \rangle$ and $(w_2)_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle x_2, r_2 \cdot v \rangle$, then $(w_1 + w_2)_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle x_1 + x_2, (r_1 + r_2) \cdot v \rangle$.

(21) Let K be a field, V be a vector space over K, v be a vector of V, X be a subspace of V, y be a vector of $X + \mathrm{Lin}(\{v\})$, and W be a subspace of $X + \mathrm{Lin}(\{v\})$. Suppose $v = y$ and $X = W$ and $v \notin X$. Let w be a vector of $X + \mathrm{Lin}(\{v\})$, x be a vector of X, and t, r be elements of K. If $w_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle x, r \cdot v \rangle$, then $(t \cdot w)_{\langle W, \mathrm{Lin}(\{y\}) \rangle} = \langle t \cdot x, t \cdot r \cdot v \rangle$.

2. Quotient Vector Space for Non Commutative Double Loop

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let W be a subspace of V. The functor $\mathrm{CosetSet}(V, W)$ yields a non empty family of subsets of V and is defined by:

(Def. 2) $\mathrm{CosetSet}(V, W) = \{A : A \text{ ranges over cosets of } W\}$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let W be a subspace of V. The functor $\mathrm{addCoset}(V, W)$ yields a binary operation on $\mathrm{CosetSet}(V, W)$ and is defined as follows:

(Def. 3) For all elements A, B of $\mathrm{CosetSet}(V,W)$ and for all vectors a, b of V such that $A = a+W$ and $B = b+W$ holds $(\mathrm{addCoset}(V,W))(A, B) = a+b+W$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let W be a subspace of V. The functor $\mathrm{zeroCoset}(V,W)$ yielding an element of $\mathrm{CosetSet}(V,W)$ is defined by:

(Def. 4) $\mathrm{zeroCoset}(V,W) =$ the carrier of W.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let W be a subspace of V. The functor $\mathrm{lmultCoset}(V,W)$ yielding a function from $[\!:$the carrier of K, $\mathrm{CosetSet}(V,W)\!:]$ into $\mathrm{CosetSet}(V,W)$ is defined by the condition (Def. 5).

(Def. 5) Let z be an element of K, A be an element of $\mathrm{CosetSet}(V,W)$, and a be a vector of V. If $A = a+W$, then $(\mathrm{lmultCoset}(V,W))(z, A) = z \cdot a+W$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let W be a subspace of V. The functor ${}^{V}/_{W}$ yielding a strict Abelian add-associative right zeroed right complementable vector space-like non empty vector space structure over K is defined by the conditions (Def. 6).

(Def. 6)(i) The carrier of ${}^{V}/_{W} = \mathrm{CosetSet}(V,W)$,

(ii) the addition of ${}^{V}/_{W} = \mathrm{addCoset}(V,W)$,

(iii) the zero of ${}^{V}/_{W} = \mathrm{zeroCoset}(V,W)$, and

(iv) the left multiplication of ${}^{V}/_{W} = \mathrm{lmultCoset}(V,W)$.

We now state several propositions:

(22) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, and W be a subspace of V. Then $\mathrm{zeroCoset}(V,W) = 0_V +W$ and $0_{{}^{V}/_{W}} = \mathrm{zeroCoset}(V,W)$.

(23) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, W be a subspace of V, and w be a vector of ${}^{V}/_{W}$. Then w is a coset of W and there exists a vector v of V such that $w = v+W$.

(24) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, W be a subspace of V, and v be a vector of V. Then $v+W$ is a coset of W and $v+W$ is a vector of ${}^{V}/_{W}$.

(25) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, and W be a subspace of V. Then every coset of W is a vector of ${}^{V}/_{W}$.

(26) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, W be a subspace of V, A be a vector of ${}^{V}/_{W}$, v be a vector of V, and a be a scalar of V. If $A = v+W$, then $a \cdot A = a \cdot v+W$.

(27) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, W be a subspace of V, A_1, A_2 be vectors of ${}^{V}/_{W}$, and v_1, v_2 be vectors of V. If $A_1 = v_1 +W$ and $A_2 = v_2 +W$, then $A_1 +A_2 = v_1 +v_2 +W$.

3. Auxiliary Facts about Functionals

The following proposition is true

(28) Let K be a field, V be a vector space over K, X be a subspace of V, f_1 be a linear functional in X, v be a vector of V, and y be a vector of $X+\mathrm{Lin}(\{v\})$. Suppose $v=y$ and $v \notin X$. Let r be an element of K. Then there exists a linear functional p_1 in $X+\mathrm{Lin}(\{v\})$ such that $p_1 \restriction$the carrier of $X=f_1$ and $p_1(y)=r$.

Let K be a right zeroed non empty loop structure and let V be a non empty vector space structure over K. One can check that there exists a functional in V which is additive and 0-preserving.

Let K be an add-associative right zeroed right complementable non empty double loop structure and let V be a right zeroed non empty vector space structure over K. Observe that every functional in V which is additive is also 0-preserving.

Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure and let V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K. One can check that every functional in V which is homogeneous is also 0-preserving.

Let K be a non empty zero structure and let V be a non empty vector space structure over K. Observe that $0\mathrm{Functional}V$ is constant.

Let K be a non empty zero structure and let V be a non empty vector space structure over K. Note that there exists a functional in V which is constant.

Let K be an add-associative right zeroed right complementable non empty double loop structure, let V be a right zeroed non empty vector space structure over K, and let f be a 0-preserving functional in V. Let us observe that f is constant if and only if:

(Def. 7) $f=0\mathrm{Functional}V$.

Let K be an add-associative right zeroed right complementable non empty double loop structure and let V be a right zeroed non empty vector space structure over K. Observe that there exists a functional in V which is constant, additive, and 0-preserving.

Let K be a non empty 1-sorted structure and let V be a non empty vector space structure over K. Observe that every functional in V which is non constant is also non trivial.

Let K be a field and let V be a non trivial vector space over K. Observe that there exists a functional in V which is additive, homogeneous, non constant, and non trivial.

Let K be a field and let V be a non trivial vector space over K. One can check that every functional in V which is trivial is also constant.

Let K be a field, let V be a non trivial vector space over K, let v be a vector of V, and let W be a linear complement of $\mathrm{Lin}(\{v\})$. Let us assume that $v \neq 0_V$. The functor $\mathrm{coeffFunctional}(v,W)$ yields a non constant non trivial linear functional in V and is defined by:

(Def. 8) $(\mathrm{coeffFunctional}(v,W))(v)=\mathbf{1}_K$ and $\mathrm{coeffFunctional}(v,W)\restriction$the carrier of $W=0\mathrm{Functional}W$.

One can prove the following propositions:

(29) Let K be a field, V be a non trivial vector space over K, and f be a non constant 0-preserving functional in V. Then there exists a vector v of V such that $v \neq 0_V$ and $f(v) \neq 0_K$.

(30) Let K be a field, V be a non trivial vector space over K, v be a vector of V, a be a scalar of V, and W be a linear complement of $\mathrm{Lin}(\{v\})$. If $v \neq 0_V$, then $(\mathrm{coeffFunctional}(v,W))(a \cdot v)=a$.

(31) Let K be a field, V be a non trivial vector space over K, v, w be vectors of V, and W be a linear complement of $\mathrm{Lin}(\{v\})$. If $v \neq 0_V$ and $w \in W$, then $(\mathrm{coeffFunctional}(v,W))(w)=0_K$.

(32) Let K be a field, V be a non trivial vector space over K, v, w be vectors of V, a be a scalar of V, and W be a linear complement of $\mathrm{Lin}(\{v\})$. If $v \neq 0_V$ and $w \in W$, then $(\mathrm{coeffFunctional}(v,W))(a \cdot v+w)=a$.

(33) Let K be a non empty loop structure, V be a non empty vector space structure over K, f, g be functionals in V, and v be a vector of V. Then $(f-g)(v)=f(v)-g(v)$.

Let K be a field and let V be a non trivial vector space over K. One can verify that $\overline{V}$ is non trivial.

4. Kernel of Additive Functional. Linear Functionals in Quotient Vector Spaces

Let K be a non empty zero structure, let V be a non empty vector space structure over K, and let f be a functional in V. The functor $\ker f$ yielding a subset of V is defined by:

(Def. 9) $\ker f = \{v; v$ ranges over vectors of V: $f(v) = 0_K\}$.

Let K be a right zeroed non empty loop structure, let V be a non empty vector space structure over K, and let f be a 0-preserving functional in V. Note that $\ker f$ is non empty.

One can prove the following proposition

(34) Let K be an add-associative right zeroed right complementable associative left unital distributive non empty double loop structure, V be an add-associative right zeroed right complementable vector space-like non empty vector space structure over K, and f be a linear functional in V. Then $\ker f$ is linearly closed.

Let K be a non empty zero structure, let V be a non empty vector space structure over K, and let f be a functional in V. We say that f is degenerated if and only if:

(Def. 10) $\ker f \neq \{0_V\}$.

Let K be a non degenerated non empty double loop structure and let V be a non trivial non empty vector space structure over K. Note that every functional in V which is non degenerated and 0-preserving is also non constant.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let f be a linear functional in V. The functor $\mathrm{Ker}\, f$ yields a strict non empty subspace of V and is defined by:

(Def. 11) The carrier of $\mathrm{Ker}\, f = \ker f$.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, let W be a subspace of V, and let f be an additive functional in V. Let us assume that the carrier of $W \subseteq \ker f$. The functor ${}^{f}/_{W}$ yielding an additive functional in ${}^{V}/_{W}$ is defined as follows:

(Def. 12) For every vector A of ${}^{V}/_{W}$ and for every vector a of V such that $A = a + W$ holds $({}^{f}/_{W})(A) = f(a)$.

One can prove the following proposition

(35) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, W be a subspace of V, and f be a linear functional in V. If the carrier of $W \subseteq \ker f$, then ${}^{f}/_{W}$ is homogeneous.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let f be a linear functional in V. The functor $\mathrm{CQFunctional}\, f$ yielding a linear functional in ${}^{V}/_{\mathrm{Ker}\, f}$ is defined by:

(Def. 13) $\mathrm{CQFunctional}\, f = {}^{f}/_{\mathrm{Ker}\, f}$.

One can prove the following proposition

(36) Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, V be a vector space over K, f be a linear functional in V, A be a vector of ${}^{V}/_{\mathrm{Ker}\, f}$, and v be a vector of V. If $A = v + \mathrm{Ker}\, f$, then $\mathrm{CQFunctional}\, f(A) = f(v)$.

Let K be a field, let V be a non trivial vector space over K, and let f be a non constant linear functional in V. One can verify that $\mathrm{CQFunctional}\, f$ is non constant.

Let K be an add-associative right zeroed right complementable Abelian associative left unital distributive non empty double loop structure, let V be a vector space over K, and let f be a linear functional in V. Note that $\mathrm{CQFunctional}\, f$ is non degenerated.

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Received November 5, 2002

Published January 2, 2004