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Real Linear-Metric Space and Isometric Functions

Robert Milewski
University of Białystok

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The articles [19], [6], [27], [21], [20], [11], [1], [22], [28], [29], [16], [3], [18], [4], [5], [15], [17], [2], [7], [25], [8], [23], [13], [24], [26], [9], [14], [12], and [10] provide the notation and terminology for this paper.

1. Convex and Internal Metric Spaces

Let V be a non empty metric structure. We say that V is convex if and only if the condition (Def. 1) is satisfied.

(Def. 1) Let x, y be elements of V and r be a real number. Suppose $0 \leq r$ and $r \leq 1$. Then there exists an element z of V such that $\rho(x,z) = r \cdot \rho(x,y)$ and $\rho(z,y) = (1-r) \cdot \rho(x,y)$.

Let V be a non empty metric structure. We say that V is internal if and only if the condition (Def. 2) is satisfied.

(Def. 2) Let x, y be elements of V and p, q be real numbers. Suppose $p > 0$ and $q > 0$. Then there exists a finite sequence f of elements of the carrier of V such that

(i) $f_1 = x$,

(ii) $f_{\mathrm{len}\, f} = y$,

(iii) for every natural number i such that $1 \leq i$ and $i \leq \mathrm{len}\, f - 1$ holds $\rho(f_i, f_{i+1}) < p$, and

(iv) for every finite sequence F of elements of $\mathbb{R}$ such that $\mathrm{len}\, F = \mathrm{len}\, f - 1$ and for every natural number i such that $1 \leq i$ and $i \leq \mathrm{len}\, F$ holds $F_i = \rho(f_i, f_{i+1})$ holds $|\rho(x,y) - \sum F| < q$.

The following proposition is true

(1) Let V be a non empty metric space. Suppose V is convex. Let x, y be elements of V and p be a real number. Suppose $p > 0$. Then there exists a finite sequence f of elements of the carrier of V such that

(i) $f_1 = x$,

(ii) $f_{\mathrm{len}\, f} = y$,

(iii) for every natural number i such that $1 \leq i$ and $i \leq \mathrm{len}\, f - 1$ holds $\rho(f_i, f_{i+1}) < p$, and

(iv) for every finite sequence F of elements of $\mathbb{R}$ such that $\mathrm{len}\, F = \mathrm{len}\, f - 1$ and for every natural number i such that $1 \leq i$ and $i \leq \mathrm{len}\, F$ holds $F_i = \rho(f_i, f_{i+1})$ holds $\rho(x,y) = \sum F$.

One can verify that every non empty metric space which is convex is also internal.

Let us note that there exists a non empty metric space which is convex.

A Geometry is a Reflexive discernible symmetric triangle internal non empty metric structure.

2. Isometric Functions

Let V be a non empty metric structure and let f be a map from V into V. We say that f is isometric if and only if:

(Def. 3) $\operatorname{rng} f =$ the carrier of V and for all elements x, y of V holds $\rho(x,y) = \rho(f(x), f(y))$.

Let V be a non empty metric structure. The functor ISOM V yielding a set is defined as follows:

(Def. 4) For every set x holds $x \in$ ISOM V iff there exists a map f from V into V such that $f = x$ and f is isometric.

Let V be a non empty metric structure. Then ISOM V is a subset of (the carrier of V)$^{\text{the carrier of } V}$.

We now state the proposition

(2) Let V be a discernible Reflexive non empty metric structure and f be a map from V into V. If f is isometric, then f is one-to-one.

Let V be a discernible Reflexive non empty metric structure. Note that every map from V into V which is isometric is also one-to-one.

Let V be a non empty metric structure. One can verify that there exists a map from V into V which is isometric.

Next we state three propositions:

(3) Let V be a discernible Reflexive non empty metric structure and f be an isometric map from V into V. Then f^{-1} is isometric.

(4) For every non empty metric structure V and for all isometric maps f, g from V into V holds $f \cdot g$ is isometric.

(5) For every non empty metric structure V holds id_V is isometric.

Let V be a non empty metric structure. One can verify that ISOM V is non empty.

3. Real Linear-Metric Spaces

We introduce RLSMetrStruct which are extensions of RLS structure and metric structure and are systems

⟨ a carrier, a distance, a zero, an addition, an external multiplication ⟩,

where the carrier is a set, the distance is a function from [: the carrier, the carrier :] into $\mathbb{R}$, the zero is an element of the carrier, the addition is a binary operation on the carrier, and the external multiplication is a function from [: $\mathbb{R}$, the carrier :] into the carrier.

One can verify that there exists a RLSMetrStruct which is non empty and strict.

Let X be a non empty set, let F be a function from [: X, X :] into $\mathbb{R}$, let O be an element of X, let B be a binary operation on X, and let G be a function from [: $\mathbb{R}$, X :] into X. Observe that $\langle X, F, O, B, G \rangle$ is non empty.

Let V be a non empty RLSMetrStruct. We say that V is homogeneous if and only if:

(Def. 5) For every real number r and for all elements v, w of V holds $\rho(r \cdot v, r \cdot w) = |r| \cdot \rho(v,w)$.

Let V be a non empty RLSMetrStruct. We say that V is translatible if and only if:

(Def. 6) For all elements u, w, v of V holds $\rho(v,w) = \rho(v+u, w+u)$.

Let V be a non empty RLSMetrStruct and let v be an element of V. The functor Norm v yields a real number and is defined by:

(Def. 7) $\operatorname{Norm} v = \rho(0_V, v)$.

One can check that there exists a non empty RLSMetrStruct which is strict, Abelian, add-associative, right zeroed, right complementable, real linear space-like, Reflexive, discernible, symmetric, triangle, homogeneous, and translatible.

A RealLinearMetrSpace is an Abelian add-associative right zeroed right complementable real linear space-like Reflexive discernible symmetric triangle homogeneous translatible non empty RLSMetrStruct.

Next we state three propositions:

(6) Let V be a homogeneous Abelian add-associative right zeroed right complementable real linear space-like non empty RLSMetrStruct, r be a real number, and v be an element of V. Then $\mathrm{Norm}(r \cdot v) = |r| \cdot \mathrm{Norm}\, v$.

(7) Let V be a translatible Abelian add-associative right zeroed right complementable triangle non empty RLSMetrStruct and v, w be elements of V. Then $\mathrm{Norm}(v+w) \leq \mathrm{Norm}\, v + \mathrm{Norm}\, w$.

(8) Let V be a translatible add-associative right zeroed right complementable non empty RLSMetrStruct and v, w be elements of V. Then $\rho(v,w) = \mathrm{Norm}(w-v)$.

Let n be a natural number. The functor RLMSpace$n$ yielding a strict RealLinearMetrSpace is defined by the conditions (Def. 8).

(Def. 8)(i) The carrier of RLMSpace$n = \mathcal{R}^n$,

(ii) the distance of RLMSpace$n = \rho^n$,

(iii) the zero of RLMSpace$n = \langle \underbrace{0, \ldots, 0}_{n} \rangle$,

(iv) for all elements x, y of $\mathcal{R}^n$ holds (the addition of RLMSpace$n$)$(x, y) = x + y$, and

(v) for every element x of $\mathcal{R}^n$ and for every element r of $\mathbb{R}$ holds (the external multiplication of RLMSpace$n$)$(r, x) = r \cdot x$.

The following proposition is true

(9) For every natural number n and for every isometric map f from RLMSpace$n$ into RLMSpace$n$ holds $\mathrm{rng}\, f = \mathcal{R}^n$.

4. Groups of Isometric Functions

Let n be a natural number. The functor IsomGroup$n$ yielding a strict groupoid is defined by the conditions (Def. 9).

(Def. 9)(i) The carrier of IsomGroup$n$ = ISOM RLMSpace$n$, and

(ii) for all functions f, g such that $f \in$ ISOM RLMSpace$n$ and $g \in$ ISOM RLMSpace$n$ holds (the multiplication of IsomGroup$n$)$(f, g) = f \cdot g$.

Let n be a natural number. Observe that IsomGroup$n$ is non empty.

Let n be a natural number. One can check that IsomGroup$n$ is associative and group-like.

The following propositions are true:

(10) For every natural number n holds $1_{\mathrm{IsomGroup}\,n} = \mathrm{id}_{\mathrm{RLMSpace}\,n}$.

(11) Let n be a natural number, f be an element of IsomGroup$n$, and g be a map from RLMSpace$n$ into RLMSpace$n$. If $f = g$, then $f^{-1} = g^{-1}$.

Let n be a natural number and let G be a subgroup of IsomGroup$n$. The functor SubIsomGroupRel$G$ yielding a binary relation on the carrier of RLMSpace$n$ is defined by the condition (Def. 10).

(Def. 10) Let A, B be elements of RLMSpace$n$. Then $\langle A, B \rangle \in$ SubIsomGroupRel$G$ if and only if there exists a function f such that $f \in$ the carrier of G and $f(A) = B$.

Let n be a natural number and let G be a subgroup of IsomGroup$n$. One can verify that SubIsomGroupRel$G$ is total, symmetric, and transitive.

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