JOURNAL OF FORMALIZED MATHEMATICS
Volume 2, Released 1990, Published 2003
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Interpretation and Satisfiability in the First Order Logic

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Summary. The main notion discussed is satisfiability. Interpretation and some auxiliary concepts are also introduced.

MML Identifier: VALUAT_1.

WWW: http://mizar.org/JFM/Vol2/valuat_1.html

The articles [6], [8], [9], [2], [3], [1], [7], [5], [4], and [10] provide the notation and terminology for this paper.

In this paper i, k denote natural numbers and A, D denote non empty sets.

Let A be a set. The functor $\boldsymbol{V}(A)$ yields a set and is defined as follows:

(Def. 1) $\boldsymbol{V}(A) = A^{\mathrm{BoundVar}}$.

Let us consider A. Observe that $\boldsymbol{V}(A)$ is non empty and functional.

Next we state the proposition

(2)[1] For every set x such that x is an element of $\boldsymbol{V}(A)$ holds x is a function from BoundVar into A.

Let us consider A. Then $\boldsymbol{V}(A)$ is a non empty set of functions from BoundVar to A.

Let f be a function. We say that f is boolean-valued if and only if:

(Def. 2) $\mathrm{rng}\, f \subseteq \mathit{Boolean}$.

Let us note that there exists a function which is boolean-valued.

Let f be a boolean-valued function and let x be a set. Note that $f(x)$ is boolean.

Let A be a set. Note that every element of $\mathit{Boolean}^A$ is boolean-valued.

Let p be a boolean-valued function. The functor $\neg p$ yields a function and is defined by:

(Def. 3) $\mathrm{dom}\,\neg p = \mathrm{dom}\, p$ and for every set x such that $x \in \mathrm{dom}\, p$ holds $(\neg p)(x) = \neg p(x)$.

Let q be a boolean-valued function. The functor $p \wedge q$ yielding a function is defined by:

(Def. 4) $\mathrm{dom}(p \wedge q) = \mathrm{dom}\, p \cap \mathrm{dom}\, q$ and for every set x such that $x \in \mathrm{dom}(p \wedge q)$ holds $(p \wedge q)(x) = p(x) \wedge q(x)$.

Let us note that the functor $p \wedge q$ is commutative.

Let p be a boolean-valued function. One can verify that $\neg p$ is boolean-valued. Let q be a boolean-valued function. Observe that $p \wedge q$ is boolean-valued.

In the sequel x, y are bound variables and v, v_1 are elements of $\boldsymbol{V}(A)$.

Let us consider A and let p be an element of $\mathit{Boolean}^A$. Then $\neg p$ is an element of $\mathit{Boolean}^A$ and it can be characterized by the condition:

[1] The proposition (1) has been removed.

(Def. 5) For every element x of A holds $(\neg p)(x) = \neg p(x)$.

Let q be an element of $\mathit{Boolean}^A$. Then $p \wedge q$ is an element of $\mathit{Boolean}^A$ and it can be characterized by the condition:

(Def. 6) For every element x of A holds $(p \wedge q)(x) = p(x) \wedge q(x)$.

Let us consider A, x and let p be an element of $\mathit{Boolean}^{\boldsymbol{V}(A)}$. The functor $\bigwedge_x p$ yields an element of $\mathit{Boolean}^{\boldsymbol{V}(A)}$ and is defined by:

(Def. 7) For every v holds $(\bigwedge_x p)(v) = \mathit{Boolean}(\mathit{false} \notin \{p(v'); v'$ ranges over elements of $\boldsymbol{V}(A)$: $\bigwedge_y (x \neq y \Rightarrow v'(y) = v(y))\})$.

The following two propositions are true:

(7)[2] Let p be an element of $\mathit{Boolean}^{\boldsymbol{V}(A)}$. Then $(\bigwedge_x p)(v) = \mathit{false}$ if and only if there exists v_1 such that $p(v_1) = \mathit{false}$ and for every y such that $x \neq y$ holds $v_1(y) = v(y)$.

(8) Let p be an element of $\mathit{Boolean}^{\boldsymbol{V}(A)}$. Then $(\bigwedge_x p)(v) = \mathit{true}$ if and only if for every v_1 such that for every y such that $x \neq y$ holds $v_1(y) = v(y)$ holds $p(v_1) = \mathit{true}$.

In the sequel l_1 is a variables list of k.

Let us consider A, k, l_1, v. Then $v \cdot l_1$ is a finite sequence of elements of A and it can be characterized by the condition:

(Def. 8) $\mathrm{len}(v \cdot l_1) = k$ and for every i such that $1 \leq i$ and $i \leq k$ holds $(v \cdot l_1)(i) = v(l_1(i))$.

We introduce $v * l_1$ as a synonym of $v \cdot l_1$.

Let us consider A, k, l_1 and let r be an element of $\mathrm{Rel}(A)$. The functor $l_1 \varepsilon r$ yields an element of $\mathit{Boolean}^{\boldsymbol{V}(A)}$ and is defined by:

(Def. 9) For every element v of $\boldsymbol{V}(A)$ holds if $v * l_1 \in r$, then $(l_1 \varepsilon r)(v) = \mathit{true}$ and if $v * l_1 \notin r$, then $(l_1 \varepsilon r)(v) = \mathit{false}$.

Let us consider A, let F be a function from CQC-WFF into $\mathit{Boolean}^{\boldsymbol{V}(A)}$, and let p be an element of CQC-WFF. Then $F(p)$ is an element of $\mathit{Boolean}^{\boldsymbol{V}(A)}$.

Let us consider D. A function from PredSym into $\mathrm{Rel}(D)$ is said to be an interpretation of D if:

(Def. 10) For every element P of PredSym and for every element r of $\mathrm{Rel}(D)$ such that $\mathrm{it}(P) = r$ holds $r = \emptyset_D$ or $\mathrm{Arity}(P) = \mathrm{Arity}(r)$.

For simplicity, we adopt the following convention: p, q, t denote elements of CQC-WFF, J denotes an interpretation of A, P denotes a k-ary predicate symbol, and r denotes an element of $\mathrm{Rel}(A)$.

Let us consider A, k, J, P. Then $J(P)$ is an element of $\mathrm{Rel}(A)$.

Let us consider A, J, p. The functor $\mathrm{Valid}(p, J)$ yielding an element of $\mathit{Boolean}^{\boldsymbol{V}(A)}$ is defined by the condition (Def. 11).

(Def. 11) There exists a function F from CQC-WFF into $\mathit{Boolean}^{\boldsymbol{V}(A)}$ such that

(i) $\mathrm{Valid}(p, J) = F(p)$,

(ii) $F(\mathrm{VERUM}) = \boldsymbol{V}(A) \longmapsto \mathit{true}$, and

(iii) for all elements p, q of CQC-WFF and for every bound variable x and for every natural number k and for every variables list l_1 of k and for every k-ary predicate symbol P holds $F(P[l_1]) = l_1 \varepsilon (J(P))$ and $F(\neg p) = \neg F(p)$ and $F(p \wedge q) = F(p) \wedge F(q)$ and $F(\forall_x p) = \bigwedge_x F(p)$.

One can prove the following propositions:

[2] The propositions (3)–(6) have been removed.

(13)[3] $\mathrm{Valid}(\mathrm{VERUM}, J) = \boldsymbol{V}(A) \longmapsto \mathit{true}$.

(14) $\mathrm{Valid}(\mathrm{VERUM}, J)(v) = \mathit{true}$.

(15) $\mathrm{Valid}(P[l_1], J) = l_1 \varepsilon(J(P))$.

(16) If $p = P[l_1]$ and $r = J(P)$, then $v * l_1 \in r$ iff $\mathrm{Valid}(p, J)(v) = \mathit{true}$.

(17) If $p = P[l_1]$ and $r = J(P)$, then $v * l_1 \notin r$ iff $\mathrm{Valid}(p, J)(v) = \mathit{false}$.

(19)[4] $\mathrm{Valid}(\neg p, J) = \neg \mathrm{Valid}(p, J)$.

(20) $\mathrm{Valid}(\neg p, J)(v) = \neg \mathrm{Valid}(p, J)(v)$.

(21) $\mathrm{Valid}(p \wedge q, J) = \mathrm{Valid}(p, J) \wedge \mathrm{Valid}(q, J)$.

(22) $\mathrm{Valid}(p \wedge q, J)(v) = \mathrm{Valid}(p, J)(v) \wedge \mathrm{Valid}(q, J)(v)$.

(23) $\mathrm{Valid}(\forall_x p, J) = \bigwedge_x \mathrm{Valid}(p, J)$.

(24) $\mathrm{Valid}(p \wedge \neg p, J)(v) = \mathit{false}$.

(25) $\mathrm{Valid}(\neg(p \wedge \neg p), J)(v) = \mathit{true}$.

Let us consider A, p, J, v. The predicate $J, v \models p$ is defined as follows:

(Def. 12) $\mathrm{Valid}(p, J)(v) = \mathit{true}$.

We now state a number of propositions:

(27)[5] $J, v \models P[l_1]$ iff $(l_1 \varepsilon(J(P)))(v) = \mathit{true}$.

(28) $J, v \models \neg p$ iff $J, v \not\models p$.

(29) $J, v \models p \wedge q$ iff $J, v \models p$ and $J, v \models q$.

(30) $J, v \models \forall_x p$ iff $(\bigwedge_x \mathrm{Valid}(p, J))(v) = \mathit{true}$.

(31) $J, v \models \forall_x p$ iff for every v_1 such that for every y such that $x \neq y$ holds $v_1(y) = v(y)$ holds $\mathrm{Valid}(p, J)(v_1) = \mathit{true}$.

(32) $\mathrm{Valid}(\neg\neg p, J) = \mathrm{Valid}(p, J)$.

(33) $\mathrm{Valid}(p \wedge p, J) = \mathrm{Valid}(p, J)$.

(35)[6] $J, v \models p \Rightarrow q$ iff $\mathrm{Valid}(p, J)(v) = \mathit{false}$ or $\mathrm{Valid}(q, J)(v) = \mathit{true}$.

(36) $J, v \models p \Rightarrow q$ iff if $J, v \models p$, then $J, v \models q$.

(37) For every element p of $\mathit{Boolean}^{\boldsymbol{V}(A)}$ such that $(\bigwedge_x p)(v) = \mathit{true}$ holds $p(v) = \mathit{true}$.

Let us consider A, J, p. The predicate $J \models p$ is defined by:

(Def. 13) For every v holds $J, v \models p$.

In the sequel w denotes an element of $\boldsymbol{V}(A)$.

The scheme *Lambda Val* deals with a non empty set $\mathcal{A}$, bound variables $\mathcal{B}$, $\mathcal{C}$, and elements $\mathcal{D}$, $\mathcal{E}$ of $\boldsymbol{V}(\mathcal{A})$, and states that:

There exists an element v of $\boldsymbol{V}(\mathcal{A})$ such that for every bound variable x such that $x \neq \mathcal{B}$ holds $v(x) = \mathcal{D}(x)$ and $v(\mathcal{B}) = \mathcal{E}(\mathcal{C})$

for all values of the parameters.

Next we state three propositions:

[3] The propositions (9)–(12) have been removed.

[4] The proposition (18) has been removed.

[5] The proposition (26) has been removed.

[6] The proposition (34) has been removed.

(39)[7] If $x \notin \mathrm{snb}(p)$, then for all v, w such that for every y such that $x \neq y$ holds $w(y) = v(y)$ holds $\mathrm{Valid}(p,J)(v) = \mathrm{Valid}(p,J)(w)$.

(40) If $J, v \models p$ and $x \notin \mathrm{snb}(p)$, then for every w such that for every y such that $x \neq y$ holds $w(y) = v(y)$ holds $J, w \models p$.

(41) $J, v \models \forall_x p$ iff for every w such that for every y such that $x \neq y$ holds $w(y) = v(y)$ holds $J, w \models p$.

In the sequel s' denotes a formula.

The following propositions are true:

(42) If $x \neq y$ and $p = s'(x)$ and $q = s'(y)$, then for every v such that $v(x) = v(y)$ holds $\mathrm{Valid}(p,J)(v) = \mathrm{Valid}(q,J)(v)$.

(43) If $x \neq y$ and $x \notin \mathrm{snb}(s')$, then $x \notin \mathrm{snb}(s'(y))$.

(44) $J, v \models \mathrm{VERUM}$.

(45) $J, v \models p \wedge q \Rightarrow q \wedge p$.

(46) $J, v \models (\neg p \Rightarrow p) \Rightarrow p$.

(47) $J, v \models p \Rightarrow (\neg p \Rightarrow q)$.

(48) $J, v \models (p \Rightarrow q) \Rightarrow (\neg(q \wedge t) \Rightarrow \neg(p \wedge t))$.

(49) If $J, v \models p$ and $J, v \models p \Rightarrow q$, then $J, v \models q$.

(50) $J, v \models \forall_x p \Rightarrow p$.

(51) $J \models \mathrm{VERUM}$.

(52) $J \models p \wedge q \Rightarrow q \wedge p$.

(53) $J \models (\neg p \Rightarrow p) \Rightarrow p$.

(54) $J \models p \Rightarrow (\neg p \Rightarrow q)$.

(55) $J \models (p \Rightarrow q) \Rightarrow (\neg(q \wedge t) \Rightarrow \neg(p \wedge t))$.

(56) If $J \models p$ and $J \models p \Rightarrow q$, then $J \models q$.

(57) $J \models \forall_x p \Rightarrow p$.

(58) If $J \models p \Rightarrow q$ and $x \notin \mathrm{snb}(p)$, then $J \models p \Rightarrow \forall_x q$.

(59) For every formula s such that $p = s(x)$ and $q = s(y)$ and $x \notin \mathrm{snb}(s)$ and $J \models p$ holds $J \models q$.

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Received June 1, 1990

Published January 2, 2004