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The Urysohn Lemma

Józef Białas
Łódź University

Yatsuka Nakamura
Shinshu University
Nagano

Summary. This article is the third part of a paper proving the fundamental Urysohn Theorem concerning the existence of a real valued continuous function on a normal topological space. The paper is divided into two parts. In the first part, we describe the construction of the function solving thesis of the Urysohn Lemma. The second part contains the proof of the Urysohn Lemma in normal space and the proof of the same theorem for compact space.

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The articles [17], [19], [2], [18], [1], [20], [8], [9], [14], [12], [11], [15], [13], [7], [16], [10], [3], [4], [5], and [6] provide the notation and terminology for this paper.

Let D be a non empty subset of $\mathbb{R}$. Observe that every element of D is real.

One can prove the following proposition

(1) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let n be a natural number. Then there exists a function G from dyadic(n) into $2^{\text{the carrier of } T}$ such that $A \subseteq G(0)$ and $B = \Omega_T \setminus G(1)$ and for all elements r_1, r_2 of dyadic(n) such that $r_1 < r_2$ holds $G(r_1)$ is open and $G(r_2)$ is open and $\overline{G(r_1)} \subseteq G(r_2)$.

Let T be a non empty topological space, let A, B be subsets of T, and let n be a natural number. Let us assume that T is a T_4 space and $A \neq \emptyset$ and A is closed and B is closed and A misses B. A function from dyadic(n) into $2^{\text{the carrier of } T}$ is said to be a drizzle of A, B, n if it satisfies the conditions (Def. 1).

(Def. 1)(i) $A \subseteq \mathrm{it}(0)$,

(ii) $B = \Omega_T \setminus \mathrm{it}(1)$, and

(iii) for all elements r_1, r_2 of dyadic(n) such that $r_1 < r_2$ holds $\mathrm{it}(r_1)$ is open and $\mathrm{it}(r_2)$ is open and $\overline{\mathrm{it}(r_1)} \subseteq \mathrm{it}(r_2)$.

Next we state the proposition

(3)[1] Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let n be a natural number and G be a drizzle of A, B, n. Then there exists a drizzle F of A, B, $n+1$ such that for every element r of dyadic($n+1$) if $r \in$ dyadic(n), then $F(r) = G(r)$.

Let A, B be non empty sets, let F be a function from $\mathbb{N}$ into $A \dot{\rightarrow} B$, and let n be a natural number. Then $F(n)$ is a partial function from A to B.

One can prove the following proposition

[1] The proposition (2) has been removed.

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(4) Let T be a non empty topological space, A, B be subsets of T, and n be a natural number. Then every drizzle of A, B, n is an element of $\mathrm{DYADIC} \dot{\to} 2^{\text{the carrier of } T}$.

Let A, B be non empty sets, let F be a function from $\mathbb{N}$ into $A \dot{\to} B$, and let n be a natural number. Then $F(n)$ is an element of $A \dot{\to} B$.

We now state the proposition

(5) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Then there exists a sequence F of partial functions from DYADIC into $2^{\text{the carrier of } T}$ such that for every natural number n holds

$F(n)$ is a drizzle of A, B, n and for every element r of $\mathrm{dom} F(n)$ holds $F(n)(r) = F(n+1)(r)$.

Let T be a non empty topological space and let A, B be subsets of T. Let us assume that T is a T_4 space and $A \neq \emptyset$ and A is closed and B is closed and A misses B. A sequence of partial functions from DYADIC into $2^{\text{the carrier of } T}$ is said to be a rain of A, B if it satisfies the condition (Def. 2).

(Def. 2) Let n be a natural number. Then $\mathrm{it}(n)$ is a drizzle of A, B, n and for every element r of $\mathrm{dom}\,\mathrm{it}(n)$ holds $\mathrm{it}(n)(r) = \mathrm{it}(n+1)(r)$.

Let x be a real number. Let us assume that $x \in \mathrm{DYADIC}$. The functor $\mathrm{InfDyadic}\, x$ yielding a natural number is defined as follows:

(Def. 3) $x \in \mathrm{dyadic}(0)$ iff $\mathrm{InfDyadic}\, x = 0$ and for every natural number n such that $x \in \mathrm{dyadic}(n+1)$ and $x \notin \mathrm{dyadic}(n)$ holds $\mathrm{InfDyadic}\, x = n+1$.

We now state several propositions:

(6) For every real number x such that $x \in \mathrm{DYADIC}$ holds $x \in \mathrm{dyadic}(\mathrm{InfDyadic}\, x)$.

(7) For every real number x such that $x \in \mathrm{DYADIC}$ and for every natural number n such that $\mathrm{InfDyadic}\, x \leq n$ holds $x \in \mathrm{dyadic}(n)$.

(8) For every real number x and for every natural number n such that $x \in \mathrm{dyadic}(n)$ holds $\mathrm{InfDyadic}\, x \leq n$.

(9) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let G be a rain of A, B and x be a real number. If $x \in \mathrm{DYADIC}$, then for every natural number n holds $G(\mathrm{InfDyadic}\, x)(x) = G(\mathrm{InfDyadic}\, x + n)(x)$.

(10) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let G be a rain of A, B and x be a real number. Suppose $x \in \mathrm{DYADIC}$. Then there exists an element y of $2^{\text{the carrier of } T}$ such that for every natural number n if $x \in \mathrm{dyadic}(n)$, then $y = G(n)(x)$.

(11) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let G be a rain of A, B. Then there exists a function F from DOM into $2^{\text{the carrier of } T}$ such that for every real number x holds

(i) if $x \in \mathbb{R}_{<0}$, then $F(x) = \emptyset$,

(ii) if $x \in \mathbb{R}_{>1}$, then $F(x) =$ the carrier of T, and

(iii) if $x \in \mathrm{DYADIC}$, then for every natural number n such that $x \in \mathrm{dyadic}(n)$ holds $F(x) = G(n)(x)$.

Let T be a non empty topological space and let A, B be subsets of T. Let us assume that T is a T_4 space and $A \neq \emptyset$ and A is closed and B is closed and A misses B. Let R be a rain of A, B. The functor $\mathrm{Tempest}\, R$ yielding a function from DOM into $2^{\text{the carrier of } T}$ is defined by the condition (Def. 4).

(Def. 4) Let x be a real number such that $x \in \mathrm{DOM}$. Then

(i) if $x \in \mathbb{R}_{<0}$, then $(\mathrm{Tempest}\, R)(x) = \emptyset$,

(ii) if $x \in \mathbb{R}_{>1}$, then $(\mathrm{Tempest}\, R)(x) =$ the carrier of T, and

(iii) if $x \in \mathrm{DYADIC}$, then for every natural number n such that $x \in \mathrm{dyadic}(n)$ holds $(\mathrm{Tempest}\, R)(x) = R(n)(x)$.

Let X be a non empty set, let T be a topological space, let F be a function from X into $2^{\text{the carrier of } T}$, and let x be an element of X. Then $F(x)$ is a subset of T.

One can prove the following three propositions:

(12) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let G be a rain of A, B, r be a real number, and C be a subset of T. If $C = (\text{Tempest}\,G)(r)$ and $r \in \text{DOM}$, then C is open.

(13) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let G be a rain of A, B and r_1, r_2 be real numbers. Suppose $r_1 \in$ DOM and $r_2 \in$ DOM and $r_1 < r_2$. Let C be a subset of T. If $C = (\text{Tempest}\,G)(r_1)$, then $\overline{C} \subseteq (\text{Tempest}\,G)(r_2)$.

(14) Let T be a non empty topological space, A, B be subsets of T, G be a rain of A, B, and p be a point of T. Then there exists a subset R of $\overline{\mathbb{R}}$ such that for every set x holds $x \in R$ if and only if the following conditions are satisfied:

(i) $x \in$ DYADIC, and

(ii) for every real number s such that $s = x$ holds $p \notin (\text{Tempest}\,G)(s)$.

Let T be a non empty topological space, let A, B be subsets of T, let R be a rain of A, B, and let p be a point of T. The functor Rainbow(p,R) yielding a subset of $\overline{\mathbb{R}}$ is defined by:

(Def. 5) For every set x holds $x \in \text{Rainbow}(p,R)$ iff $x \in$ DYADIC and for every real number s such that $s = x$ holds $p \notin (\text{Tempest}\,R)(s)$.

Let T, S be non empty topological spaces, let F be a function from the carrier of T into the carrier of S, and let p be a point of T. Then $F(p)$ is a point of S.

We now state two propositions:

(15) Let T be a non empty topological space, A, B be subsets of T, G be a rain of A, B, and p be a point of T. Then Rainbow$(p,G) \subseteq$ DYADIC.

(16) Let T be a non empty topological space, A, B be subsets of T, and R be a rain of A, B. Then there exists a map F from T into $\mathbb{R}^{\mathbf{1}}$ such that for every point p of T holds

if Rainbow$(p,R) = \emptyset$, then $F(p) = 0$ and for every non empty subset S of $\overline{\mathbb{R}}$ such that $S =$ Rainbow(p,R) holds $F(p) = \sup S$.

Let T be a non empty topological space, let A, B be subsets of T, and let R be a rain of A, B. The functor Thunder$R$ yielding a map from T into $\mathbb{R}^{\mathbf{1}}$ is defined by the condition (Def. 6).

(Def. 6) Let p be a point of T. Then if Rainbow$(p,R) = \emptyset$, then $(\text{Thunder}\,R)(p) = 0$ and for every non empty subset S of $\overline{\mathbb{R}}$ such that $S = \text{Rainbow}(p,R)$ holds $(\text{Thunder}\,R)(p) = \sup S$.

Let T be a non empty topological space, let F be a map from T into $\mathbb{R}^{\mathbf{1}}$, and let p be a point of T. Then $F(p)$ is a real number.

The following propositions are true:

(17) Let T be a non empty topological space, A, B be subsets of T, G be a rain of A, B, p be a point of T, and S be a non empty subset of $\overline{\mathbb{R}}$. Suppose $S = \text{Rainbow}(p,G)$. Let ℓ_1 be an extended real number. If $\ell_1 = 1$, then $0_{\overline{\mathbb{R}}} \leq \sup S$ and $\sup S \leq \ell_1$.

(18) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let G be a rain of A, B, r be an element of DOM, and p be a point of T. If $(\text{Thunder}\,G)(p) < r$, then $p \in (\text{Tempest}\,G)(r)$.

(19) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let G be a rain of A, B and r be a real number. Suppose $r \in \text{DYADIC} \cup \mathbb{R}_{>1}$ and $0 < r$. Let p be a point of T. If $p \in (\text{Tempest}\,G)(r)$, then $(\text{Thunder}\,G)(p) \leq r$.

(20) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let G be a rain of A, B, n be a natural number, and r_1 be an element of DOM. If $0 < r_1$, then for every point p of T such that $r_1 < (\mathrm{Thunder}\,G)(p)$ holds $p \notin (\mathrm{Tempest}\,G)(r_1)$.

(21) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Let G be a rain of A, B. Then

(i) $\mathrm{Thunder}\,G$ is continuous, and

(ii) for every point x of T holds $0 \leq (\mathrm{Thunder}\,G)(x)$ and $(\mathrm{Thunder}\,G)(x) \leq 1$ and if $x \in A$, then $(\mathrm{Thunder}\,G)(x) = 0$ and if $x \in B$, then $(\mathrm{Thunder}\,G)(x) = 1$.

(22) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose $A \neq \emptyset$ and A misses B. Then there exists a map F from T into $\mathbb{R}^{\mathbf{1}}$ such that

(i) F is continuous, and

(ii) for every point x of T holds $0 \leq F(x)$ and $F(x) \leq 1$ and if $x \in A$, then $F(x) = 0$ and if $x \in B$, then $F(x) = 1$.

(23) Let T be a non empty T_4 topological space and A, B be closed subsets of T. Suppose A misses B. Then there exists a map F from T into $\mathbb{R}^{\mathbf{1}}$ such that

(i) F is continuous, and

(ii) for every point x of T holds $0 \leq F(x)$ and $F(x) \leq 1$ and if $x \in A$, then $F(x) = 0$ and if $x \in B$, then $F(x) = 1$.

(24) Let T be a non empty T_2 compact topological space and A, B be closed subsets of T. Suppose A misses B. Then there exists a map F from T into $\mathbb{R}^{\mathbf{1}}$ such that

(i) F is continuous, and

(ii) for every point x of T holds $0 \leq F(x)$ and $F(x) \leq 1$ and if $x \in A$, then $F(x) = 0$ and if $x \in B$, then $F(x) = 1$.

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