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Little Bezout Theorem (Factor Theorem)[1]

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Summary. We present a formalization of the factor theorem for univariate polynomials, also called the (little) Bezout theorem: Let r belong to a commutative ring L and $p(x)$ be a polynomial over L. Then $x-r$ divides $p(x)$ iff $p(r)=0$. We also prove some consequences of this theorem like that any non zero polynomial of degree n over an algebraically closed integral domain has n (non necessarily distinct) roots.

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The articles [27], [37], [31], [8], [2], [26], [32], [15], [20], [38], [6], [7], [3], [9], [36], [33], [24], [23], [11], [21], [16], [19], [17], [18], [1], [12], [34], [28], [22], [10], [35], [4], [25], [39], [13], [29], [14], [30], and [5] provide the notation and terminology for this paper.

1. Preliminaries

One can prove the following propositions:

(1) For every natural number n holds n is non empty iff $n=1$ or $n>1$.

(2) Let f be a finite sequence of elements of $\mathbb{N}$. Suppose that for every natural number i such that $i \in \operatorname{dom} f$ holds $f(i) \neq 0$. Then $\sum f = \operatorname{len} f$ if and only if $f = \operatorname{len} f \mapsto 1$.

The scheme *IndFinSeq0* deals with a finite sequence $\mathcal{A}$ and a binary predicate $\mathcal{P}$, and states that:

For every natural number i such that $1 \leq i$ and $i \leq \operatorname{len} \mathcal{A}$ holds $\mathcal{P}[i, \mathcal{A}(i)]$

provided the following conditions are met:

- $\mathcal{P}[1, \mathcal{A}(1)]$, and
- For every natural number i such that $1 \leq i$ and $i < \operatorname{len} \mathcal{A}$ holds if $\mathcal{P}[i, \mathcal{A}(i)]$, then $\mathcal{P}[i+1, \mathcal{A}(i+1)]$.

The following proposition is true

(3) Let L be an add-associative right zeroed right complementable non empty loop structure and r be a finite sequence of elements of L. Suppose $\operatorname{len} r \geq 2$ and for every natural number k such that $2 < k$ and $k \in \operatorname{dom} r$ holds $r(k) = 0_L$. Then $\sum r = r_1 + r_2$.

2. Canonical Ordering of a Finite Set

Let A be a finite set. The functor $\operatorname{canFS}(A)$ yields a finite sequence of elements of A and is defined by the conditions (Def. 1).

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(Def. 1)(i) $\operatorname{len}\operatorname{canFS}(A) = \operatorname{card}A$, and

(ii) there exists a finite sequence f such that $\operatorname{len}f = \operatorname{card}A$ and $f(1) = \langle \operatorname{choose}(A), A \setminus \{\operatorname{choose}(A)\} \rangle$ or $\operatorname{card}A = 0$ and for every natural number i such that $1 \leq i$ and $i < \operatorname{card}A$ and for every set x such that $f(i) = x$ holds $f(i+1) = \langle \operatorname{choose}(x_{\mathbf{2}}), x_{\mathbf{2}} \setminus \{\operatorname{choose}(x_{\mathbf{2}})\} \rangle$ and for every natural number i such that $i \in \operatorname{dom}\operatorname{canFS}(A)$ holds $(\operatorname{canFS}(A))(i) = f(i)_{\mathbf{1}}$.

Next we state four propositions:

(4) For every finite set A holds $\operatorname{canFS}(A)$ is one-to-one.

(5) For every finite set A holds $\operatorname{rng}\operatorname{canFS}(A) = A$.

(6) For every set a holds $\operatorname{canFS}(\{a\}) = \langle a \rangle$.

(7) For every finite set A holds $(\operatorname{canFS}(A))^{-1}$ is a function from A into $\operatorname{Seg}\operatorname{card}A$.

3. More about Bags

Let X be a set, let S be a finite subset of X, and let n be a natural number. The functor $(S,n)-\operatorname{bag}$ yielding an element of $\operatorname{Bags}X$ is defined by:

(Def. 2) $(S,n)-\operatorname{bag} = \operatorname{EmptyBag}X {+\cdot} (S \longmapsto n)$.

The following propositions are true:

(8) Let X be a set, S be a finite subset of X, n be a natural number, and i be a set. If $i \notin S$, then $((S,n)-\operatorname{bag})(i) = 0$.

(9) Let X be a set, S be a finite subset of X, n be a natural number, and i be a set. If $i \in S$, then $((S,n)-\operatorname{bag})(i) = n$.

(10) For every set X and for every finite subset S of X and for every natural number n such that $n \neq 0$ holds $\operatorname{support}(S,n)-\operatorname{bag} = S$.

(11) Let X be a set, S be a finite subset of X, and n be a natural number. If S is empty or $n = 0$, then $(S,n)-\operatorname{bag} = \operatorname{EmptyBag}X$.

(12) Let X be a set, S, T be finite subsets of X, and n be a natural number. If S misses T, then $(S \cup T,n)-\operatorname{bag} = (S,n)-\operatorname{bag} + (T,n)-\operatorname{bag}$.

Let A be a set and let b be a bag of A. The functor $\operatorname{degree}(b)$ yielding a natural number is defined as follows:

(Def. 3) There exists a finite sequence f of elements of $\mathbb{N}$ such that $\operatorname{degree}(b) = \sum f$ and $f = b \cdot \operatorname{canFS}(\operatorname{support}b)$.

Next we state several propositions:

(13) For every set A and for every bag b of A holds $b = \operatorname{EmptyBag}A$ iff $\operatorname{degree}(b) = 0$.

(14) Let A be a set, S be a finite subset of A, and b be a bag of A. Then $S = \operatorname{support}b$ and $\operatorname{degree}(b) = \operatorname{card}S$ if and only if $b = (S,1)-\operatorname{bag}$.

(15) Let A be a set, S be a finite subset of A, and b be a bag of A. Suppose $\operatorname{support}b \subseteq S$. Then there exists a finite sequence f of elements of $\mathbb{N}$ such that $f = b \cdot \operatorname{canFS}(S)$ and $\operatorname{degree}(b) = \sum f$.

(16) For every set A and for all bags b, b_1, b_2 of A such that $b = b_1 + b_2$ holds $\operatorname{degree}(b) = \operatorname{degree}(b_1) + \operatorname{degree}(b_2)$.

(17) Let L be an associative commutative unital non empty groupoid, f, g be finite sequences of elements of L, and p be a permutation of $\operatorname{dom}f$. If $g = f \cdot p$, then $\prod g = \prod f$.

4. More on Polynomials

Let L be a non empty zero structure and let p be a polynomial of L. We say that p is non-zero if and only if:

(Def. 4) $p \neq \mathbf{0}.L$.

The following proposition is true

(18) For every non empty zero structure L and for every polynomial p of L holds p is non-zero iff $\operatorname{len} p > 0$.

Let L be a non trivial non empty zero structure. One can verify that there exists a polynomial of L which is non-zero.

Let L be a non degenerated non empty multiplicative loop with zero structure and let x be an element of L. Observe that $\langle_0 x, \mathbf{1}_L\rangle$ is non-zero.

We now state three propositions:

(19) For every non empty zero structure L and for every polynomial p of L such that $\operatorname{len} p > 0$ holds $p(\operatorname{len} p -' 1) \neq 0_L$.

(20) Let L be a non empty zero structure and p be an algebraic sequence of L. If $\operatorname{len} p = 1$, then $p = \langle_0 p(0)\rangle$ and $p(0) \neq 0_L$.

(21) Let L be an add-associative right zeroed right complementable right distributive non empty double loop structure and p be a polynomial of L. Then $p * \mathbf{0}.L = \mathbf{0}.L$.

One can verify that there exists a well unital non empty double loop structure which is algebraic-closed, add-associative, right zeroed, right complementable, Abelian, commutative, associative, distributive, integral domain-like, and non degenerated.

Next we state the proposition

(22) Let L be an add-associative right zeroed right complementable distributive integral domain-like non empty double loop structure and p, q be polynomials of L. If $p * q = \mathbf{0}.L$, then $p = \mathbf{0}.L$ or $q = \mathbf{0}.L$.

Let L be an add-associative right zeroed right complementable distributive integral domain-like non empty double loop structure. Note that Polynom-Ring L is integral domain-like.

Let L be an integral domain and let p, q be non-zero polynomials of L. Observe that $p * q$ is non-zero.

Next we state a number of propositions:

(23) For every non degenerated commutative ring L and for all polynomials p, q of L holds $\operatorname{Roots} p \cup \operatorname{Roots} q \subseteq \operatorname{Roots}(p * q)$.

(24) For every integral domain L and for all polynomials p, q of L holds $\operatorname{Roots}(p * q) = \operatorname{Roots} p \cup \operatorname{Roots} q$.

(25) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure, p be a polynomial of L, and p_1 be an element of Polynom-Ring L. If $p = p_1$, then $-p = -p_1$.

(26) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure, p, q be polynomials of L, and p_1, q_1 be elements of Polynom-Ring L. If $p = p_1$ and $q = q_1$, then $p - q = p_1 - q_1$.

(27) Let L be an Abelian add-associative right zeroed right complementable distributive non empty double loop structure and p, q, r be polynomials of L. Then $p * q - p * r = p * (q - r)$.

(28) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure and p, q be polynomials of L. If $p - q = \mathbf{0}.L$, then $p = q$.

(29) Let L be an Abelian add-associative right zeroed right complementable distributive integral domain-like non empty double loop structure and p, q, r be polynomials of L. If $p \neq \mathbf{0}.L$ and $p * q = p * r$, then $q = r$.

(30) Let L be an integral domain, n be a natural number, and p be a polynomial of L. If $p \neq \mathbf{0}.L$, then $p^n \neq \mathbf{0}.L$.

(31) For every commutative ring L and for all natural numbers i, j and for every polynomial p of L holds $p^i * p^j = p^{i+j}$.

(32) For every non empty multiplicative loop with zero structure L holds $\mathbf{1}.L = \langle_0 \mathbf{1}_L \rangle$.

(33) Let L be an add-associative right zeroed right complementable right unital right distributive non empty double loop structure and p be a polynomial of L. Then $p * \langle_0 \mathbf{1}_L \rangle = p$.

(34) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure and p, q be polynomials of L. If $\operatorname{len} p = 0$ or $\operatorname{len} q = 0$, then $\operatorname{len}(p * q) = 0$.

(35) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure and p, q be polynomials of L. If $p * q$ is non-zero, then p is non-zero and q is non-zero.

(36) Let L be an add-associative right zeroed right complementable distributive commutative associative left unital non empty double loop structure and p, q be polynomials of L. If $p(\operatorname{len} p -' 1) \cdot q(\operatorname{len} q -' 1) \neq 0_L$, then $0 < \operatorname{len}(p * q)$.

(37) Let L be an add-associative right zeroed right complementable distributive commutative associative left unital integral domain-like non empty double loop structure and p, q be polynomials of L. If $1 < \operatorname{len} p$ and $1 < \operatorname{len} q$, then $\operatorname{len} p < \operatorname{len}(p * q)$.

(38) Let L be an add-associative right zeroed right complementable left distributive non empty double loop structure, a, b be elements of L, and p be a polynomial of L. Then $(\langle_0 a, b\rangle * p)(0) = a \cdot p(0)$ and for every natural number i holds $(\langle_0 a, b\rangle * p)(i+1) = a \cdot p(i+1) + b \cdot p(i)$.

(39) Let L be an add-associative right zeroed right complementable distributive well unital commutative associative non degenerated non empty double loop structure, r be an element of L, and q be a non-zero polynomial of L. Then $\operatorname{len}(\langle_0 r, \mathbf{1}_L\rangle * q) = \operatorname{len} q + 1$.

(40) Let L be a non degenerated commutative ring, x be an element of L, and i be a natural number. Then $\operatorname{len}(\langle_0 x, \mathbf{1}_L\rangle^i) = i + 1$.

Let L be a non degenerated commutative ring, let x be an element of L, and let n be a natural number. One can verify that $\langle_0 x, \mathbf{1}_L\rangle^n$ is non-zero.

We now state two propositions:

(41) Let L be a non degenerated commutative ring, x be an element of L, q be a non-zero polynomial of L, and i be a natural number. Then $\operatorname{len}(\langle_0 x, \mathbf{1}_L\rangle^i * q) = i + \operatorname{len} q$.

(42) Let L be an add-associative right zeroed right complementable distributive well unital commutative associative non degenerated non empty double loop structure, r be an element of L, and p, q be polynomials of L. If $p = \langle_0 r, \mathbf{1}_L\rangle * q$ and $p(\operatorname{len} p -' 1) = \mathbf{1}_L$, then $q(\operatorname{len} q -' 1) = \mathbf{1}_L$.

5. Little Bezout Theorem

Let L be a non empty zero structure, let p be a polynomial of L, and let n be a natural number. The functor $\text{poly\_shift}(p, n)$ yielding a polynomial of L is defined as follows:

(Def. 5) For every natural number i holds $(\text{poly\_shift}(p, n))(i) = p(n + i)$.

One can prove the following propositions:

(43) For every non empty zero structure L and for every polynomial p of L holds $\mathrm{poly\_shift}(p,0) = p$.

(44) Let L be a non empty zero structure, n be a natural number, and p be a polynomial of L. If $n \geq \mathrm{len}\, p$, then $\mathrm{poly\_shift}(p,n) = \mathbf{0}.L$.

(45) Let L be a non degenerated non empty multiplicative loop with zero structure, n be a natural number, and p be a polynomial of L. If $n \leq \mathrm{len}\, p$, then $\mathrm{len}\,\mathrm{poly\_shift}(p,n) + n = \mathrm{len}\, p$.

(46) Let L be a non degenerated commutative ring, x be an element of L, n be a natural number, and p be a polynomial of L. If $n < \mathrm{len}\, p$, then $\mathrm{eval}(\mathrm{poly\_shift}(p,n),x) = x \cdot \mathrm{eval}(\mathrm{poly\_shift}(p,n+1),x) + p(n)$.

(47) For every non degenerated commutative ring L and for every polynomial p of L such that $\mathrm{len}\, p = 1$ holds $\mathrm{Roots}\, p = \emptyset$.

Let L be a non degenerated commutative ring, let r be an element of L, and let p be a polynomial of L. Let us assume that r is a root of p. The functor $\mathrm{poly\_quotient}(p,r)$ yielding a polynomial of L is defined as follows:

(Def. 6)(i) $\mathrm{len}\,\mathrm{poly\_quotient}(p,r) + 1 = \mathrm{len}\, p$ and for every natural number i holds $(\mathrm{poly\_quotient}(p,r))(i) = \mathrm{eval}(\mathrm{poly\_shift}(p,i+1),r)$ if $\mathrm{len}\, p > 0$,

(ii) $\mathrm{poly\_quotient}(p,r) = \mathbf{0}.L$, otherwise.

One can prove the following propositions:

(48) Let L be a non degenerated commutative ring, r be an element of L, and p be a non-zero polynomial of L. If r is a root of p, then $\mathrm{len}\,\mathrm{poly\_quotient}(p,r) > 0$.

(49) Let L be an add-associative right zeroed right complementable left distributive well unital non empty double loop structure and x be an element of L. Then $\mathrm{Roots}\langle {}_0{-x}, \mathbf{1}_L\rangle = \{x\}$.

(50) Let L be a non trivial commutative ring, x be an element of L, and p, q be polynomials of L. If $p = \langle {}_0{-x}, \mathbf{1}_L\rangle * q$, then x is a root of p.

(51) Let L be a non degenerated commutative ring, r be an element of L, and p be a polynomial of L. If r is a root of p, then $p = \langle {}_0{-r}, \mathbf{1}_L\rangle * \mathrm{poly\_quotient}(p,r)$.

(52) Let L be a non degenerated commutative ring, r be an element of L, and p, q be polynomials of L. If $p = \langle {}_0{-r}, \mathbf{1}_L\rangle * q$, then r is a root of p.

6. Polynomials Defined by Roots

Let L be an integral domain and let p be a non-zero polynomial of L. Note that $\mathrm{Roots}\, p$ is finite.

Let L be a non degenerated commutative ring, let x be an element of L, and let p be a non-zero polynomial of L. The functor $\mathrm{multiplicity}(p,x)$ yields a natural number and is defined by the condition (Def. 7).

(Def. 7) There exists a finite non empty subset F of $\mathbb{N}$ such that $F = \{k; k$ ranges over natural numbers: $\bigvee_{q:\text{polynomial of } L}\ p = \langle {}_0{-x}, \mathbf{1}_L\rangle^k * q\}$ and $\mathrm{multiplicity}(p,x) = \max F$.

One can prove the following two propositions:

(53) Let L be a non degenerated commutative ring, p be a non-zero polynomial of L, and x be an element of L. Then x is a root of p if and only if $\mathrm{multiplicity}(p,x) \geq 1$.

(54) For every non degenerated commutative ring L and for every element x of L holds $\mathrm{multiplicity}(\langle {}_0{-x}, \mathbf{1}_L\rangle, x) = 1$.

Let L be an integral domain and let p be a non-zero polynomial of L. The functor $\mathrm{BRoots}(p)$ yielding a bag of the carrier of L is defined as follows:

(Def. 8) support BRoots(p) = Roots p and for every element x of L holds $(\text{BRoots}(p))(x) = \text{multiplicity}(p,x)$.

One can prove the following propositions:

(55) For every integral domain L and for every element x of L holds $\text{BRoots}(\langle_0 -x, \mathbf{1}_L\rangle) = (\{x\},1)-\text{bag}$.

(56) Let L be an integral domain, x be an element of L, and p, q be non-zero polynomials of L. Then $\text{multiplicity}(p*q,x) = \text{multiplicity}(p,x) + \text{multiplicity}(q,x)$.

(57) For every integral domain L and for all non-zero polynomials p, q of L holds $\text{BRoots}(p*q) = \text{BRoots}(p) + \text{BRoots}(q)$.

(58) For every integral domain L and for every non-zero polynomial p of L such that $\text{len}\, p = 1$ holds $\text{degree}(\text{BRoots}(p)) = 0$.

(59) For every integral domain L and for every element x of L and for every natural number n holds $\text{degree}(\text{BRoots}(\langle_0 -x, \mathbf{1}_L\rangle^n)) = n$.

(60) For every algebraic-closed integral domain L and for every non-zero polynomial p of L holds $\text{degree}(\text{BRoots}(p)) = \text{len}\, p -' 1$.

Let L be an add-associative right zeroed right complementable distributive non empty double loop structure, let c be an element of L, and let n be a natural number. The functor fpoly_mult_root(c,n) yields a finite sequence of elements of Polynom-Ring L and is defined by:

(Def. 9) len fpoly_mult_root$(c,n) = n$ and for every natural number i such that $i \in$ dom fpoly_mult_root(c,n) holds $(\text{fpoly\_mult\_root}(c,n))(i) = \langle_0 -c, \mathbf{1}_L\rangle$.

Let L be an add-associative right zeroed right complementable distributive non empty double loop structure and let b be a bag of the carrier of L. The functor poly_with_roots(b) yielding a polynomial of L is defined by the condition (Def. 10).

(Def. 10) There exists a finite sequence f of elements of (the carrier of Polynom-Ring $L)^*$ and there exists a finite sequence s of elements of L such that $\text{len}\, f = \text{card support}\, b$ and $s = \text{canFS}(\text{support}\, b)$ and for every natural number i such that $i \in \text{dom}\, f$ holds $f(i) = \text{fpoly\_mult\_root}(s_i, b(s_i))$ and $\text{poly\_with\_roots}(b) = \prod \text{Flat}(f)$.

The following propositions are true:

(61) Let L be an Abelian add-associative right zeroed right complementable commutative distributive right unital non empty double loop structure. Then poly_with_roots(EmptyBag (the carrier of L)) $= \langle_0 \mathbf{1}_L\rangle$.

(62) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure and c be an element of L. Then $\text{poly\_with\_roots}((\{c\},1)-\text{bag}) = \langle_0 -c, \mathbf{1}_L\rangle$.

(63) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure, b be a bag of the carrier of L, f be a finite sequence of elements of (the carrier of Polynom-Ring $L)^*$, and s be a finite sequence of elements of L. Suppose $\text{len}\, f = \text{card support}\, b$ and $s = \text{canFS}(\text{support}\, b)$ and for every natural number i such that $i \in \text{dom}\, f$ holds $f(i) = \text{fpoly\_mult\_root}(s_i, b(s_i))$. Then $\text{len Flat}(f) = \text{degree}(b)$.

(64) Let L be an add-associative right zeroed right complementable distributive non empty double loop structure, b be a bag of the carrier of L, f be a finite sequence of elements of (the carrier of Polynom-Ring $L)^*$, s be a finite sequence of elements of L, and c be an element of L such that $\text{len}\, f = \text{card support}\, b$ and $s = \text{canFS}(\text{support}\, b)$ and for every natural number i such that $i \in \text{dom}\, f$ holds $f(i) = \text{fpoly\_mult\_root}(s_i, b(s_i))$. Then

(i) if $c \in \text{support}\, b$, then $\text{card}(\text{Flat}(f)^{-1}(\{\langle_0 -c, \mathbf{1}_L\rangle\})) = b(c)$, and

(ii) if $c \notin \text{support}\, b$, then $\text{card}(\text{Flat}(f)^{-1}(\{\langle_0 -c, \mathbf{1}_L\rangle\})) = 0$.

(65) For every commutative ring L and for all bags b_1, b_2 of the carrier of L holds $\mathrm{poly\_with\_roots}(b_1+b_2) = \mathrm{poly\_with\_roots}(b_1) * \mathrm{poly\_with\_roots}(b_2)$.

(66) Let L be an algebraic-closed integral domain and p be a non-zero polynomial of L. If $p(\mathrm{len}\, p -' 1) = \mathbf{1}_L$, then $p = \mathrm{poly\_with\_roots}(\mathrm{BRoots}(p))$.

(67) Let L be a commutative ring, s be a non empty finite subset of L, and f be a finite sequence of elements of Polynom-Ring$L$. Suppose $\mathrm{len}\, f = \overline{\overline{s}}$ and for every natural number i and for every element c of L such that $i \in \mathrm{dom}\, f$ and $c = (\mathrm{canFS}(s))(i)$ holds $f(i) = \langle {}_0{-c}, \mathbf{1}_L \rangle$. Then $\mathrm{poly\_with\_roots}((s,1)-\mathrm{bag}) = \prod f$.

(68) Let L be a non trivial commutative ring, s be a non empty finite subset of L, x be an element of L, and f be a finite sequence of elements of L. Suppose $\mathrm{len}\, f = \overline{\overline{s}}$ and for every natural number i and for every element c of L such that $i \in \mathrm{dom}\, f$ and $c = (\mathrm{canFS}(s))(i)$ holds $f(i) = \mathrm{eval}(\langle {}_0{-c}, \mathbf{1}_L \rangle, x)$. Then $\mathrm{eval}(\mathrm{poly\_with\_roots}((s,1)-\mathrm{bag}), x) = \prod f$.

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