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Primitive Roots of Unity and Cyclotomic Polynomials[1]

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Summary. We present a formalization of roots of unity, define cyclotomic polynomials and demonstrate the relationship between cyclotomic polynomials and unital polynomials.

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The articles [33], [42], [34], [13], [9], [15], [35], [18], [2], [27], [36], [17], [25], [5], [43], [6], [7], [4], [16], [11], [40], [37], [8], [10], [28], [12], [26], [19], [20], [23], [21], [22], [24], [1], [41], [44], [29], [14], [38], [32], [3], [39], [31], [45], and [30] provide the notation and terminology for this paper.

1. Preliminaries

The following proposition is true

(1) For every natural number n holds $n = 0$ or $n = 1$ or $n \geq 2$.

The scheme *Comp Ind NE* concerns a unary predicate $\mathcal{P}$, and states that:

For every non empty natural number k holds $\mathcal{P}[k]$

provided the following condition is satisfied:

- For every non empty natural number k such that for every non empty natural number n such that $n < k$ holds $\mathcal{P}[n]$ holds $\mathcal{P}[k]$.

We now state the proposition

(2) For every finite sequence f such that $1 \leq \operatorname{len} f$ holds $f \restriction \operatorname{Seg} 1 = \langle f(1) \rangle$.

One can prove the following propositions:

(3) Let f be a finite sequence of elements of $\mathbb{C}_{\rm F}$ and g be a finite sequence of elements of $\mathbb{R}$. Suppose $\operatorname{len} f = \operatorname{len} g$ and for every natural number i such that $i \in \operatorname{dom} f$ holds $|f_i| = g(i)$. Then $|\prod f| = \prod g$.

(4) Let s be a non empty finite subset of $\mathbb{C}_{\rm F}$, x be an element of $\mathbb{C}_{\rm F}$, and r be a finite sequence of elements of $\mathbb{R}$. Suppose $\operatorname{len} r = \operatorname{card} s$ and for every natural number i and for every element c of $\mathbb{C}_{\rm F}$ such that $i \in \operatorname{dom} r$ and $c = (\operatorname{canFS}(s))(i)$ holds $r(i) = |x - c|$. Then $|\operatorname{eval}(\text{poly\_with\_roots}((s, 1) - \text{bag}), x)| = \prod r$.

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(5) Let f be a finite sequence of elements of $\mathbb{C}_{\mathrm{F}}$. Suppose that for every natural number i such that $i \in \mathrm{dom}\, f$ holds $f(i)$ is integer. Then $\sum f$ is integer.

(6) For every real number r there exists an element z of $\mathbb{C}$ such that $z = r$ and $z = r + 0i$.

(7) For all elements x, y of $\mathbb{C}_{\mathrm{F}}$ and for all real numbers r_1, r_2 such that $r_1 = x$ and $r_2 = y$ holds $r_1 \cdot r_2 = x \cdot y$ and $r_1 + r_2 = x + y$.

(8) Let q be a real number. Suppose q is an integer and $q > 0$. Let r be an element of $\mathbb{C}_{\mathrm{F}}$. If $|r| = 1$ and $r \neq 1 + 0i_{\mathbb{C}_{\mathrm{F}}}$, then $|(q + 0i_{\mathbb{C}_{\mathrm{F}}}) - r| > q - 1$.

(9) Let p_1 be a non empty finite sequence of elements of $\mathbb{R}$ and x be a real number. Suppose $x \geq 1$ and for every natural number i such that $i \in \mathrm{dom}\, p_1$ holds $p_1(i) > x$. Then $\prod p_1 > x$.

(10) For every natural number n holds $\mathbf{1}_{\mathbb{C}_{\mathrm{F}}} = \mathrm{power}_{\mathbb{C}_{\mathrm{F}}}(\mathbf{1}_{\mathbb{C}_{\mathrm{F}}}, n)$.

(11) Let n be a non empty natural number and i be a natural number. Then $\cos(\frac{2 \cdot \pi \cdot i}{n}) = \cos(\frac{2 \cdot \pi \cdot (i \bmod n)}{n})$ and $\sin(\frac{2 \cdot \pi \cdot i}{n}) = \sin(\frac{2 \cdot \pi \cdot (i \bmod n)}{n})$.

(12) For every non empty natural number n and for every natural number i holds $\cos(\frac{2 \cdot \pi \cdot i}{n}) + \sin(\frac{2 \cdot \pi \cdot i}{n}) i_{\mathbb{C}_{\mathrm{F}}} = \cos(\frac{2 \cdot \pi \cdot (i \bmod n)}{n}) + \sin(\frac{2 \cdot \pi \cdot (i \bmod n)}{n}) i_{\mathbb{C}_{\mathrm{F}}}$.

(13) Let n be a non empty natural number and i, j be natural numbers. Then $(\cos(\frac{2 \cdot \pi \cdot i}{n}) + \sin(\frac{2 \cdot \pi \cdot i}{n}) i_{\mathbb{C}_{\mathrm{F}}}) \cdot (\cos(\frac{2 \cdot \pi \cdot j}{n}) + \sin(\frac{2 \cdot \pi \cdot j}{n}) i_{\mathbb{C}_{\mathrm{F}}}) = \cos(\frac{2 \cdot \pi \cdot ((i+j) \bmod n)}{n}) + \sin(\frac{2 \cdot \pi \cdot ((i+j) \bmod n)}{n}) i_{\mathbb{C}_{\mathrm{F}}}$.

(14) Let L be a unital associative non empty groupoid, x be an element of L, and n, m be natural numbers. Then $\mathrm{power}_L(x, n \cdot m) = \mathrm{power}_L(\mathrm{power}_L(x, n), m)$.

(15) For every natural number n and for every element x of $\mathbb{C}_{\mathrm{F}}$ such that x is an integer holds $\mathrm{power}_{\mathbb{C}_{\mathrm{F}}}(x, n)$ is an integer.

(16) Let F be a finite sequence of elements of $\mathbb{C}_{\mathrm{F}}$. Suppose that for every natural number i such that $i \in \mathrm{dom} F$ holds $F(i)$ is an integer. Then $\sum F$ is an integer.

(17) For every real number a such that $0 \leq a$ and $a < 2 \cdot \pi$ and $\cos a = 1$ holds $a = 0$.

Let us note that there exists a field which is finite and there exists a skew field which is finite.

2. Multiplicative Group of a Skew Field

Let R be a skew field. The functor $\mathrm{MultGroup}(R)$ yields a strict group and is defined by the conditions (Def. 1).

(Def. 1)(i) The carrier of $\mathrm{MultGroup}(R) =$ (the carrier of R) $\setminus \{0_R\}$, and

(ii) the multiplication of $\mathrm{MultGroup}(R) =$ (the multiplication of R)$\restriction$[:the carrier of $\mathrm{MultGroup}(R)$, the carrier of $\mathrm{MultGroup}(R)$:].

One can prove the following three propositions:

(18) For every skew field R holds the carrier of $R =$ (the carrier of $\mathrm{MultGroup}(R)$) $\cup \{0_R\}$.

(19) Let R be a skew field, a, b be elements of R, and c, d be elements of $\mathrm{MultGroup}(R)$. If $a = c$ and $b = d$, then $c \cdot d = a \cdot b$.

(20) For every skew field R holds $\mathbf{1}_R = 1_{\mathrm{MultGroup}(R)}$.

Let R be a finite skew field. Note that $\mathrm{MultGroup}(R)$ is finite.
Next we state three propositions:

(21) For every finite skew field R holds $\mathrm{ord}(\mathrm{MultGroup}(R)) = \mathrm{card}$(the carrier of R) $- 1$.

(22) For every skew field R and for every set s such that $s \in$ the carrier of $\mathrm{MultGroup}(R)$ holds $s \in$ the carrier of R.

(23) For every skew field R holds the carrier of $\mathrm{MultGroup}(R) \subseteq$ the carrier of R.

3. Roots of Unity

Let n be a non empty natural number. The functor $n-\mathrm{roots\_of\_1}$ yields a subset of $\mathbb{C}_{\mathrm{F}}$ and is defined by:

(Def. 2) $n-\mathrm{roots\_of\_1} = \{x; x$ ranges over elements of $\mathbb{C}_{\mathrm{F}}$: x is a complex root of n, $\mathbf{1}_{\mathbb{C}_{\mathrm{F}}}\}$.

Next we state several propositions:

(24) Let n be a non empty natural number and x be an element of $\mathbb{C}_{\mathrm{F}}$. Then $x \in n-\mathrm{roots\_of\_1}$ if and only if x is a complex root of n, $\mathbf{1}_{\mathbb{C}_{\mathrm{F}}}$.

(25) For every non empty natural number n holds $\mathbf{1}_{\mathbb{C}_{\mathrm{F}}} \in n-\mathrm{roots\_of\_1}$.

(26) For every non empty natural number n and for every element x of $\mathbb{C}_{\mathrm{F}}$ such that $x \in n-\mathrm{roots\_of\_1}$ holds $|x| = 1$.

(27) Let n be a non empty natural number and x be an element of $\mathbb{C}_{\mathrm{F}}$. Then $x \in n-\mathrm{roots\_of\_1}$ if and only if there exists a natural number k such that $x = \cos(\frac{2\cdot\pi\cdot k}{n}) + \sin(\frac{2\cdot\pi\cdot k}{n}) i_{\mathbb{C}_{\mathrm{F}}}$.

(28) For every non empty natural number n and for all elements x, y of $\mathbb{C}$ such that $x \in n-\mathrm{roots\_of\_1}$ and $y \in n-\mathrm{roots\_of\_1}$ holds $x \cdot y \in n-\mathrm{roots\_of\_1}$.

(29) For every non empty natural number n holds $n-\mathrm{roots\_of\_1} = \{\cos(\frac{2\cdot\pi\cdot k}{n}) + \sin(\frac{2\cdot\pi\cdot k}{n}) i_{\mathbb{C}_{\mathrm{F}}}; k$ ranges over natural numbers: $k < n\}$.

(30) For every non empty natural number n holds $\overline{\overline{n-\mathrm{roots\_of\_1}}} = n$.

Let n be a non empty natural number. Observe that $n-\mathrm{roots\_of\_1}$ is non empty and $n-\mathrm{roots\_of\_1}$ is finite.

One can prove the following propositions:

(31) For all non empty natural numbers n, n_1 such that $n_1 \mid n$ holds $n_1-\mathrm{roots\_of\_1} \subseteq n-\mathrm{roots\_of\_1}$.

(32) Let R be a skew field, x be an element of $\mathrm{MultGroup}(R)$, and y be an element of R. If $y = x$, then for every natural number k holds $\mathrm{power}_{\mathrm{MultGroup}(R)}(x, k) = \mathrm{power}_R(y, k)$.

(33) For every non empty natural number n and for every element x of $\mathrm{MultGroup}(\mathbb{C}_{\mathrm{F}})$ such that $x \in n-\mathrm{roots\_of\_1}$ holds x is not of order 0.

(34) Let n be a non empty natural number, k be a natural number, and x be an element of $\mathrm{MultGroup}(\mathbb{C}_{\mathrm{F}})$. If $x = \cos(\frac{2\cdot\pi\cdot k}{n}) + \sin(\frac{2\cdot\pi\cdot k}{n}) i_{\mathbb{C}_{\mathrm{F}}}$, then $\mathrm{ord}(x) = n \div (k \gcd n)$.

(35) For every non empty natural number n holds $n-\mathrm{roots\_of\_1} \subseteq$ the carrier of $\mathrm{MultGroup}(\mathbb{C}_{\mathrm{F}})$.

(36) For every non empty natural number n there exists an element x of $\mathrm{MultGroup}(\mathbb{C}_{\mathrm{F}})$ such that $\mathrm{ord}(x) = n$.

(37) For every non empty natural number n and for every element x of $\mathrm{MultGroup}(\mathbb{C}_{\mathrm{F}})$ holds $\mathrm{ord}(x) \mid n$ iff $x \in n-\mathrm{roots\_of\_1}$.

(38) For every non empty natural number n holds $n-\mathrm{roots\_of\_1} = \{x; x$ ranges over elements of $\mathrm{MultGroup}(\mathbb{C}_{\mathrm{F}})$: $\mathrm{ord}(x) \mid n\}$.

(39) Let n be a non empty natural number and x be a set. Then $x \in n-\mathrm{roots\_of\_1}$ if and only if there exists an element y of $\mathrm{MultGroup}(\mathbb{C}_{\mathrm{F}})$ such that $x = y$ and $\mathrm{ord}(y) \mid n$.

Let n be a non empty natural number. The functor $n-\mathrm{th\_roots\_of\_1}$ yields a strict group and is defined by:

(Def. 3) The carrier of $n-$ th_roots_of_1 $= n-$ roots_of_1 and the multiplication of $n-$ th_roots_of_1 $=$ (the multiplication of $\mathbb{C}_{\mathrm{F}}$) $\upharpoonright [\!: n-\text{roots\_of\_1}, n-\text{roots\_of\_1} :\!]$.

The following proposition is true

(40) For every non empty natural number n holds $n-$ th_roots_of_1 is a subgroup of MultGroup($\mathbb{C}_{\mathrm{F}}$).

4. The Unital Polynomial $x^n - 1$

Let n be a non empty natural number and let L be a left unital non empty double loop structure. The functor unital_poly(L,n) yielding a polynomial of L is defined as follows:

(Def. 4) unital_poly$(L,n) = \mathbf{0}.L +\cdot (0, -\mathbf{1}_L) +\cdot (n, \mathbf{1}_L)$.

We now state four propositions:

(41) unital_poly$(\mathbb{C}_{\mathrm{F}}, 1) = \langle {}_0 - \mathbf{1}_{\mathbb{C}_{\mathrm{F}}}, \mathbf{1}_{\mathbb{C}_{\mathrm{F}}} \rangle$.

(42) Let L be a left unital non empty double loop structure and n be a non empty natural number. Then (unital_poly$(L,n))(0) = -\mathbf{1}_L$ and (unital_poly$(L,n))(n) = \mathbf{1}_L$.

(43) Let L be a left unital non empty double loop structure, n be a non empty natural number, and i be a natural number. If $i \neq 0$ and $i \neq n$, then (unital_poly$(L,n))(i) = 0_L$.

(44) Let L be a non degenerated left unital non empty double loop structure and n be a non empty natural number. Then len unital_poly$(L,n) = n+1$.

Let L be a non degenerated left unital non empty double loop structure and let n be a non empty natural number. Observe that unital_poly(L,n) is non-zero.

We now state several propositions:

(45) For every non empty natural number n and for every element x of $\mathbb{C}_{\mathrm{F}}$ holds eval(unital_poly$(\mathbb{C}_{\mathrm{F}},n),x) = \mathrm{power}_{\mathbb{C}_{\mathrm{F}}}(x, n) - 1$.

(46) For every non empty natural number n holds Roots unital_poly$(\mathbb{C}_{\mathrm{F}},n) = n-$ roots_of_1 .

(47) Let n be a natural number and z be an element of $\mathbb{C}_{\mathrm{F}}$. Suppose z is a real number. Then there exists a real number x such that $x = z$ and $\mathrm{power}_{\mathbb{C}_{\mathrm{F}}}(z, n) = x^n$.

(48) Let n be a non empty natural number and x be a real number. Then there exists an element y of $\mathbb{C}_{\mathrm{F}}$ such that $y = x$ and eval(unital_poly$(\mathbb{C}_{\mathrm{F}},n),y) = x^n - 1$.

(49) For every non empty natural number n holds BRoots(unital_poly$(\mathbb{C}_{\mathrm{F}},n)) = (n-$ roots_of_1$, 1) -$ bag .

(50) For every non empty natural number n holds unital_poly$(\mathbb{C}_{\mathrm{F}},n) =$ poly_with_roots($(n-$ roots_of_1$, 1) -$ bag).

Let i be an integer and let n be a natural number. Then i^n is an integer.

Next we state the proposition

(51) For every non empty natural number n and for every element i of $\mathbb{C}_{\mathrm{F}}$ such that i is an integer holds eval(unital_poly$(\mathbb{C}_{\mathrm{F}},n),i)$ is an integer.

5. Cyclotomic Polynomials

Let d be a non empty natural number. The functor cyclotomic_poly(d) yields a polynomial of $\mathbb{C}_{\mathrm{F}}$ and is defined as follows:

(Def. 5) There exists a non empty finite subset s of $\mathbb{C}_{\mathrm{F}}$ such that $s = \{y; y$ ranges over elements of MultGroup($\mathbb{C}_{\mathrm{F}}$): $\mathrm{ord}(y) = d\}$ and $\mathrm{cyclotomic\_poly}(d) = \mathrm{poly\_with\_roots}((s,1)-\mathrm{bag})$.

Next we state a number of propositions:

(52) $\mathrm{cyclotomic\_poly}(1) = \langle_0 -\mathbf{1}_{\mathbb{C}_{\mathrm{F}}}, \mathbf{1}_{\mathbb{C}_{\mathrm{F}}}\rangle$.

(53) Let n be a non empty natural number and f be a finite sequence of elements of the carrier of Polynom-Ring($\mathbb{C}_{\mathrm{F}}$). Suppose $\mathrm{len}\, f = n$ and for every non empty natural number i such that $i \in \mathrm{dom}\, f$ holds if $i \nmid n$, then $f(i) = \langle_0 \mathbf{1}_{\mathbb{C}_{\mathrm{F}}}\rangle$ and if $i \mid n$, then $f(i) = \mathrm{cyclotomic\_poly}(i)$. Then $\mathrm{unital\_poly}(\mathbb{C}_{\mathrm{F}}, n) = \prod f$.

(54) Let n be a non empty natural number. Then there exists a finite sequence f of elements of the carrier of Polynom-Ring($\mathbb{C}_{\mathrm{F}}$) and there exists a polynomial p of $\mathbb{C}_{\mathrm{F}}$ such that

(i) $p = \prod f$,

(ii) $\mathrm{dom}\, f = \mathrm{Seg}\, n$,

(iii) for every non empty natural number i such that $i \in \mathrm{Seg}\, n$ holds if $i \nmid n$ or $i = n$, then $f(i) = \langle_0 \mathbf{1}_{\mathbb{C}_{\mathrm{F}}}\rangle$ and if $i \mid n$ and $i \neq n$, then $f(i) = \mathrm{cyclotomic\_poly}(i)$, and

(iv) $\mathrm{unital\_poly}(\mathbb{C}_{\mathrm{F}}, n) = \mathrm{cyclotomic\_poly}(n) * p$.

(55) For every non empty natural number d and for every natural number i holds $(\mathrm{cyclotomic\_poly}(d))(0) = 1$ or $(\mathrm{cyclotomic\_poly}(d))(0) = -1$ but $(\mathrm{cyclotomic\_poly}(d))(i)$ is integer.

(56) For every non empty natural number d and for every element z of $\mathbb{C}_{\mathrm{F}}$ such that z is an integer holds $\mathrm{eval}(\mathrm{cyclotomic\_poly}(d), z)$ is an integer.

(57) Let n, n_1 be non empty natural numbers, f be a finite sequence of elements of the carrier of Polynom-Ring($\mathbb{C}_{\mathrm{F}}$), and s be a finite subset of $\mathbb{C}_{\mathrm{F}}$. Suppose that

(i) $s = \{y; y$ ranges over elements of MultGroup($\mathbb{C}_{\mathrm{F}}$): $\mathrm{ord}(y) \mid n \wedge \mathrm{ord}(y) \nmid n_1 \wedge \mathrm{ord}(y) \neq n\}$,

(ii) $\mathrm{dom}\, f = \mathrm{Seg}\, n$, and

(iii) for every non empty natural number i such that $i \in \mathrm{dom}\, f$ holds if $i \nmid n$ or $i \mid n_1$ or $i = n$, then $f(i) = \langle_0 \mathbf{1}_{\mathbb{C}_{\mathrm{F}}}\rangle$ and if $i \mid n$ and $i \nmid n_1$ and $i \neq n$, then $f(i) = \mathrm{cyclotomic\_poly}(i)$.

Then $\prod f = \mathrm{poly\_with\_roots}((s,1)-\mathrm{bag})$.

(58) Let n, n_1 be non empty natural numbers. Suppose $n_1 < n$ and $n_1 \mid n$. Then there exists a finite sequence f of elements of the carrier of Polynom-Ring($\mathbb{C}_{\mathrm{F}}$) and there exists a polynomial p of $\mathbb{C}_{\mathrm{F}}$ such that

(i) $p = \prod f$,

(ii) $\mathrm{dom}\, f = \mathrm{Seg}\, n$,

(iii) for every non empty natural number i such that $i \in \mathrm{Seg}\, n$ holds if $i \nmid n$ or $i \mid n_1$ or $i = n$, then $f(i) = \langle_0 \mathbf{1}_{\mathbb{C}_{\mathrm{F}}}\rangle$ and if $i \mid n$ and $i \nmid n_1$ and $i \neq n$, then $f(i) = \mathrm{cyclotomic\_poly}(i)$, and

(iv) $\mathrm{unital\_poly}(\mathbb{C}_{\mathrm{F}}, n) = \mathrm{unital\_poly}(\mathbb{C}_{\mathrm{F}}, n_1) * \mathrm{cyclotomic\_poly}(n) * p$.

(59) Let i be an integer, c be an element of $\mathbb{C}_{\mathrm{F}}$, f be a finite sequence of elements of the carrier of Polynom-Ring($\mathbb{C}_{\mathrm{F}}$), and p be a polynomial of $\mathbb{C}_{\mathrm{F}}$. Suppose $p = \prod f$ and $c = i$ and for every non empty natural number i such that $i \in \mathrm{dom}\, f$ holds $f(i) = \langle_0 \mathbf{1}_{\mathbb{C}_{\mathrm{F}}}\rangle$ or $f(i) = \mathrm{cyclotomic\_poly}(i)$. Then $\mathrm{eval}(p, c)$ is integer.

(60) Let n be a non empty natural number, j, k, q be integers, and q_1 be an element of $\mathbb{C}_{\mathrm{F}}$. If $q_1 = q$ and $j = \mathrm{eval}(\mathrm{cyclotomic\_poly}(n), q_1)$ and $k = \mathrm{eval}(\mathrm{unital\_poly}(\mathbb{C}_{\mathrm{F}}, n), q_1)$, then $j \mid k$.

(61) Let n, n_1 be non empty natural numbers and q be an integer. Suppose $n_1 < n$ and $n_1 \mid n$. Let q_1 be an element of c_1. Suppose $q_1 = q$. Let j, k, l be integers. If $j = \mathrm{eval}(\mathrm{cyclotomic\_poly}(n), q_1)$ and $k = \mathrm{eval}(\mathrm{unital\_poly}(\mathbb{C}_{\mathrm{F}}, n), q_1)$ and $l = \mathrm{eval}(\mathrm{unital\_poly}(\mathbb{C}_{\mathrm{F}}, n_1), q_1)$, then $j \mid k \div l$, where $c_1 =$ the carrier of $\mathbb{C}_{\mathrm{F}}$.

(62) Let n, q be non empty natural numbers and q_1 be an element of $\mathbb{C}_{\mathrm{F}}$. If $q_1 = q$, then for every integer j such that $j = \mathrm{eval}(\mathrm{cyclotomic\_poly}(n), q_1)$ holds $j \mid q^n - 1$.

(63) Let n, n_1, q be non empty natural numbers. Suppose $n_1 < n$ and $n_1 \mid n$. Let q_1 be an element of $\mathbb{C}_{\mathrm{F}}$. If $q_1 = q$, then for every integer j such that $j = \mathrm{eval}(\mathrm{cyclotomic\_poly}(n), q_1)$ holds $j \mid (q^n - 1) \div (q^{n_1} - 1)$.

(64) Let n be a non empty natural number. Suppose $1 < n$. Let q be a natural number. Suppose $1 < q$. Let q_1 be an element of $\mathbb{C}_{\mathrm{F}}$. If $q_1 = q$, then for every integer i such that $i = \mathrm{eval}(\mathrm{cyclotomic\_poly}(n), q_1)$ holds $|i| > q - 1$.

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