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Lebesgue's Covering Lemma, Uniform Continuity and Segmentation of Arcs

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Summary. For mappings from a metric space to a metric space, a notion of uniform continuity is defined. If we introduce natural topologies to the metric spaces, a uniformly continuous function becomes continuous. On the other hand, if the domain is compact, a continuous function is uniformly continuous. For this proof, Lebesgue's covering lemma is also proved. An arc, which is homeomorphic to [0,1], can be divided into small segments, as small as one wishes.

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The articles [20], [24], [21], [16], [13], [1], [2], [22], [19], [18], [25], [3], [5], [6], [23], [11], [17], [8], [7], [10], [9], [12], [14], [4], and [15] provide the notation and terminology for this paper.

1. Lebesgue's Covering Lemma

We follow the rules: s, s_1, s_2, t, r, r_1, r_2 are real numbers and n, m are natural numbers.

We now state two propositions:

(1) $t-r-(t-s) = -r+s$ and $t-r-(t-s) = s-r$.

(2) For every r such that $r > 0$ there exists a natural number n such that $n > 0$ and $\frac{1}{n} < r$.

Let X, Y be non empty metric structures and let f be a map from X into Y. We say that f is uniformly continuous if and only if:

(Def. 1) For every r such that $0 < r$ there exists s such that $0 < s$ and for all elements u_1, u_2 of X such that $\rho(u_1,u_2) < s$ holds $\rho(f_{u_1}, f_{u_2}) < r$.

We now state several propositions:

(3) Let X be a non empty topological space, M be a non empty metric space, and f be a map from X into $M_{\rm top}$. Suppose f is continuous. Let r be a real number, u be an element of the carrier of M, and P be a subset of $M_{\rm top}$. If $P = \mathrm{Ball}(u,r)$, then $f^{-1}(P)$ is open.

(4) Let N, M be non empty metric spaces and f be a map from $N_{\rm top}$ into $M_{\rm top}$. Suppose that for every real number r and for every element u of the carrier of N and for every element u_1 of M such that $r > 0$ and $u_1 = f(u)$ there exists a real number s such that $s > 0$ and for every element w of N and for every element w_1 of M such that $w_1 = f(w)$ and $\rho(u,w) < s$ holds $\rho(u_1,w_1) < r$. Then f is continuous.

(5) Let N, M be non empty metric spaces and f be a map from $N_{\rm top}$ into $M_{\rm top}$. Suppose f is continuous. Let r be a real number, u be an element of the carrier of N, and u_1 be an element of M. Suppose $r > 0$ and $u_1 = f(u)$. Then there exists s such that $s > 0$ and for every element w of N and for every element w_1 of M such that $w_1 = f(w)$ and $\rho(u,w) < s$ holds $\rho(u_1,w_1) < r$.

(6) Let N, M be non empty metric spaces, f be a map from N into M, and g be a map from $N_{\rm top}$ into $M_{\rm top}$. If $f = g$ and f is uniformly continuous, then g is continuous.

(7) Let N be a non empty metric space and G be a family of subsets of $N_{\rm top}$. Suppose G is a cover of $N_{\rm top}$ and open and $N_{\rm top}$ is compact. Then there exists r such that $r > 0$ and for all elements w_1, w_2 of N such that $\rho(w_1,w_2) < r$ there exists a subset G_1 of $N_{\rm top}$ such that $w_1 \in G_1$ and $w_2 \in G_1$ and $G_1 \in G$.

2. Uniformity of Continuous Functions on Compact Spaces

Next we state three propositions:

(8) Let N, M be non empty metric spaces, f be a map from N into M, and g be a map from $N_{\rm top}$ into $M_{\rm top}$. Suppose $g = f$ and $N_{\rm top}$ is compact and g is continuous. Then f is uniformly continuous.

(9) Let g be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\rm T}$ and f be a map from $[0, 1]_{\rm M}$ into $\mathcal{E}^n$. If g is continuous and $f = g$, then f is uniformly continuous.

(10) Let P be a subset of $\mathcal{E}^n_{\rm T}$, Q be a non empty subset of $\mathcal{E}^n$, g be a map from $\mathbb{I}$ into $(\mathcal{E}^n_{\rm T})\restriction P$, and f be a map from $[0, 1]_{\rm M}$ into $\mathcal{E}^n\restriction Q$. If $P = Q$ and g is continuous and $f = g$, then f is uniformly continuous.

3. Segmentation of Arcs

Next we state four propositions:

(11) For every map g from $\mathbb{I}$ into $\mathcal{E}^n_{\rm T}$ there exists a map f from $[0, 1]_{\rm M}$ into $\mathcal{E}^n$ such that $f = g$.

(12) Let r be a real number. Suppose $r \geq 0$. Then $\lceil r\rceil \geq 0$ and $\lfloor r\rfloor \geq 0$ and $\lceil r\rceil$ is a natural number and $\lfloor r\rfloor$ is a natural number.

(13) For all r, s holds $|r-s| = |s-r|$.

(14) For all r_1, r_2, s_1, s_2 such that $r_1 \in [s_1,s_2]$ and $r_2 \in [s_1,s_2]$ holds $|r_1-r_2| \leq s_2-s_1$.

Let I_1 be a finite sequence of elements of $\mathbb{R}$. We say that I_1 is decreasing if and only if:

(Def. 2) For all n, m such that $n \in \operatorname{dom} I_1$ and $m \in \operatorname{dom} I_1$ and $n < m$ holds $I_1(n) > I_1(m)$.

We now state two propositions:

(15) Let e be a real number, g be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\rm T}$, and p_1, p_2 be elements of $\mathcal{E}^n_{\rm T}$. Suppose $e > 0$ and g is continuous and one-to-one and $g(0) = p_1$ and $g(1) = p_2$. Then there exists a finite sequence h of elements of $\mathbb{R}$ such that

(i) $h(1) = 0$,

(ii) $h(\operatorname{len} h) = 1$,

(iii) $5 \leq \operatorname{len} h$,

(iv) $\operatorname{rng} h \subseteq$ the carrier of $\mathbb{I}$,

(v) h is increasing, and

(vi) for every natural number i and for every subset Q of $\mathbb{I}$ and for every subset W of $\mathcal{E}^n$ such that $1 \leq i$ and $i < \operatorname{len} h$ and $Q = [h_i, h_{i+1}]$ and $W = g^\circ Q$ holds $\emptyset W < e$.

(16) Let e be a real number, g be a map from $\mathbb{I}$ into $\mathcal{E}^n_{\mathrm{T}}$, and p_1, p_2 be elements of $\mathcal{E}^n_{\mathrm{T}}$. Suppose $e > 0$ and g is continuous and one-to-one and $g(0) = p_1$ and $g(1) = p_2$. Then there exists a finite sequence h of elements of $\mathbb{R}$ such that

(i) $h(1) = 1$,

(ii) $h(\mathrm{len}\, h) = 0$,

(iii) $5 \leq \mathrm{len}\, h$,

(iv) $\mathrm{rng}\, h \subseteq$ the carrier of $\mathbb{I}$,

(v) h is decreasing, and

(vi) for every natural number i and for every subset Q of $\mathbb{I}$ and for every subset W of $\mathcal{E}^n$ such that $1 \leq i$ and $i < \mathrm{len}\, h$ and $Q = [h_{i+1}, h_i]$ and $W = g^{\circ} Q$ holds $\varnothing W < e$.

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