JOURNAL OF FORMALIZED MATHEMATICS
Volume 7, Released 1995, Published 2003
Inst. of Computer Science, Univ. of Białystok

On the Lattice of Subalgebras of a Universal Algebra

Mirosław Jan Paszek
Warsaw University
Białystok

MML Identifier: UNIALG_3.

WWW: http://mizar.org/JFM/Vol7/unialg_3.html

The articles [10], [6], [12], [9], [13], [4], [5], [2], [7], [14], [1], [8], [11], and [3] provide the notation and terminology for this paper.

In this paper U_0 is a universal algebra and o is an operation of U_0.

Let us consider U_0. Family of subalgebras of U_0 is defined by:

(Def. 1) For every set U_1 such that $U_1 \in$ it holds U_1 is a subalgebra of U_0.

Let us consider U_0. Observe that there exists a family of subalgebras of U_0 which is non empty.

Let us consider U_0. Then $\mathrm{Sub}(U_0)$ is a non empty family of subalgebras of U_0. Let U_2 be a non empty family of subalgebras of U_0. We see that the element of U_2 is a subalgebra of U_0.

Let us consider U_0. Then $\bigsqcup_{(U_0)}$ is a binary operation on $\mathrm{Sub}(U_0)$. Then $\bigsqcap_{(U_0)}$ is a binary operation on $\mathrm{Sub}(U_0)$.

Let us consider U_0 and let u be an element of $\mathrm{Sub}(U_0)$. The functor $\overline{u}$ yielding a subset of U_0 is defined as follows:

(Def. 2) There exists a subalgebra U_1 of U_0 such that $u = U_1$ and $\overline{u} =$ the carrier of U_1.

Let us consider U_0. The functor $\mathrm{Carr}(U_0)$ yielding a function from $\mathrm{Sub}(U_0)$ into $2^{\text{the carrier of } U_0}$ is defined by:

(Def. 3) For every element u of $\mathrm{Sub}(U_0)$ holds $(\mathrm{Carr}(U_0))(u) = \overline{u}$.

Next we state several propositions:

(1) For every set u holds $u \in \mathrm{Sub}(U_0)$ iff there exists a strict subalgebra U_1 of U_0 such that $u = U_1$.

(2) For every non empty subset H of U_0 and for every o such that $\mathrm{arity}\, o = 0$ holds H is closed on o iff $o(\emptyset) \in H$.

(3) For every subalgebra U_1 of U_0 holds the carrier of $U_1 \subseteq$ the carrier of U_0.

(4) For every non empty subset H of U_0 and for every o such that H is closed on o and $\mathrm{arity}\, o = 0$ holds $o_H = o$.

(5) $\mathrm{Constants}(U_0) = \{o(\emptyset); o$ ranges over operation of U_0: $\mathrm{arity}\, o = 0\}$.

(6) For every universal algebra U_0 with constants and for every subalgebra U_1 of U_0 holds $\mathrm{Constants}(U_0) = \mathrm{Constants}(U_1)$.

Let U_0 be a universal algebra with constants. Note that every subalgebra of U_0 has constants. The following proposition is true

(7) For every universal algebra U_0 with constants and for all subalgebras U_1, U_3 of U_0 holds $\mathrm{Constants}(U_1) = \mathrm{Constants}(U_3)$.

Let us consider U_0. Then $\mathrm{Carr}(U_0)$ is a function from $\mathrm{Sub}(U_0)$ into $2^{\text{the carrier of } U_0}$ and it can be characterized by the condition:

(Def. 4) For every element u of $\mathrm{Sub}(U_0)$ and for every subalgebra U_1 of U_0 such that $u = U_1$ holds $(\mathrm{Carr}(U_0))(u) =$ the carrier of U_1.

The following propositions are true:

(8) For every strict subalgebra H of U_0 and for every element u of U_0 holds $u \in (\mathrm{Carr}(U_0))(H)$ iff $u \in H$.

(9) For every non empty subset H of $\mathrm{Sub}(U_0)$ holds $(\mathrm{Carr}(U_0))^\circ H$ is non empty.

(10) For every universal algebra U_0 with constants and for every strict subalgebra U_1 of U_0 holds $\mathrm{Constants}(U_0) \subseteq (\mathrm{Carr}(U_0))(U_1)$.

(11) Let U_0 be a universal algebra with constants, U_1 be a subalgebra of U_0, and a be a set. If a is an element of $\mathrm{Constants}(U_0)$, then $a \in$ the carrier of U_1.

(12) For every universal algebra U_0 with constants and for every non empty subset H of $\mathrm{Sub}(U_0)$ holds $\bigcap((\mathrm{Carr}(U_0))^\circ H)$ is a non empty subset of U_0.

(13) For every universal algebra U_0 with constants holds the carrier of the lattice of subalgebras of $U_0 = \mathrm{Sub}(U_0)$.

(14) Let U_0 be a universal algebra with constants, H be a non empty subset of $\mathrm{Sub}(U_0)$, and S be a non empty subset of U_0. If $S = \bigcap((\mathrm{Carr}(U_0))^\circ H)$, then S is operations closed.

Let U_0 be a strict universal algebra with constants and let H be a non empty subset of $\mathrm{Sub}(U_0)$. The functor $\bigcap H$ yields a strict subalgebra of U_0 and is defined as follows:

(Def. 5) The carrier of $\bigcap H = \bigcap((\mathrm{Carr}(U_0))^\circ H)$.

The following propositions are true:

(15) Let U_0 be a universal algebra with constants, l_1, l_2 be elements of the lattice of subalgebras of U_0, and U_1, U_3 be strict subalgebras of U_0. If $l_1 = U_1$ and $l_2 = U_3$, then $l_1 \sqcup l_2 = U_1 \sqcup U_3$.

(16) Let U_0 be a universal algebra with constants, l_1, l_2 be elements of the lattice of subalgebras of U_0, and U_1, U_3 be strict subalgebras of U_0. If $l_1 = U_1$ and $l_2 = U_3$, then $l_1 \sqcap l_2 = U_1 \cap U_3$.

(17) Let U_0 be a universal algebra with constants, l_1, l_2 be elements of the lattice of subalgebras of U_0, and U_1, U_3 be strict subalgebras of U_0. Suppose $l_1 = U_1$ and $l_2 = U_3$. Then $l_1 \sqsubseteq l_2$ if and only if the carrier of $U_1 \subseteq$ the carrier of U_3.

(18) Let U_0 be a universal algebra with constants, l_1, l_2 be elements of the lattice of subalgebras of U_0, and U_1, U_3 be strict subalgebras of U_0. If $l_1 = U_1$ and $l_2 = U_3$, then $l_1 \sqsubseteq l_2$ iff U_1 is a subalgebra of U_3.

(19) For every strict universal algebra U_0 with constants holds the lattice of subalgebras of U_0 is bounded.

Let U_0 be a strict universal algebra with constants. Observe that the lattice of subalgebras of U_0 is bounded.

Next we state several propositions:

(20) For every strict universal algebra U_0 with constants and for every strict subalgebra U_1 of U_0 holds $\mathrm{Gen}^{\mathrm{UA}}(\mathrm{Constants}(U_0)) \cap U_1 = \mathrm{Gen}^{\mathrm{UA}}(\mathrm{Constants}(U_0))$.

(21) For every strict universal algebra U_0 with constants holds $\perp_{\text{the lattice of subalgebras of } U_0} = \mathrm{Gen}^{\mathrm{UA}}(\mathrm{Constants}(U_0))$.

(22) Let U_0 be a strict universal algebra with constants, U_1 be a subalgebra of U_0, and H be a subset of U_0. If $H =$ the carrier of U_0, then $\mathrm{Gen}^{\mathrm{UA}}(H) \sqcup U_1 = \mathrm{Gen}^{\mathrm{UA}}(H)$.

(23) Let U_0 be a strict universal algebra with constants and H be a subset of U_0. Suppose $H =$ the carrier of U_0. Then $\top_{\text{the lattice of subalgebras of } U_0} = \mathrm{Gen}^{\mathrm{UA}}(H)$.

(24) For every strict universal algebra U_0 with constants holds $\top_{\text{the lattice of subalgebras of } U_0} = U_0$.

(25) For every strict universal algebra U_0 with constants holds the lattice of subalgebras of U_0 is complete.

Let U_4, U_5 be universal algebras with constants and let F be a function from the carrier of U_4 into the carrier of U_5. The functor $\mathrm{FuncLatt}(F)$ yielding a function from the carrier of the lattice of subalgebras of U_4 into the carrier of the lattice of subalgebras of U_5 is defined by:

(Def. 6) For every strict subalgebra U_1 of U_4 and for every subset H of U_5 such that $H = F^{\circ}$(the carrier of U_1) holds $(\mathrm{FuncLatt}(F))(U_1) = \mathrm{Gen}^{\mathrm{UA}}(H)$.

One can prove the following proposition

(26) Let U_0 be a strict universal algebra with constants and F be a function from the carrier of U_0 into the carrier of U_0. Suppose $F = \mathrm{id}_{\text{the carrier of } U_0}$. Then $\mathrm{FuncLatt}(F) = \mathrm{id}_{\text{the carrier of the lattice of subalgebras of } U_0}$.

REFERENCES

[1] Grzegorz Bancerek. Complete lattices. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/lattice3.html`.

[2] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[3] Ewa Burakowska. Subalgebras of the universal algebra. Lattices of subalgebras. *Journal of Formalized Mathematics*, 5, 1993. `http://mizar.org/JFM/Vol5/unialg_2.html`.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_2.html`.

[6] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[7] Czesław Byliński. Finite sequences and tuples of elements of a non-empty sets. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/finseq_2.html`.

[8] Jarosław Kotowicz, Beata Madras, and Małgorzata Korolkiewicz. Basic notation of universal algebra. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/unialg_1.html`.

[9] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/setfam_1.html`.

[10] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[11] Wojciech A. Trybulec. Vectors in real linear space. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/rlvect_1.html`.

[12] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[13] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[14] Stanisław Żukowski. Introduction to lattice theory. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/lattices.html.

Received May 23, 1995

Published January 2, 2004