JOURNAL OF FORMALIZED MATHEMATICS
Volume 6, Released 1994, Published 2003
Inst. of Computer Science, Univ. of Białystok

Maximal Kolmogorov Subspaces of a Topological Space as Stone Retracts of the Ambient Space[1]

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Summary. Let X be a topological space. X is said to be T_0-*space* (or *Kolmogorov space*) provided for every pair of distinct points $x, y \in X$ there exists an open subset of X containing exactly one of these points (see [1], [9], [5]). Such spaces and subspaces were investigated in Mizar formalism in [8]. A Kolmogorov subspace X_0 of a topological space X is said to be *maximal* provided for every Kolmogorov subspace Y of X if X_0 is subspace of Y then the topological structures of Y and X_0 are the same.

M.H. Stone proved in [11] that every topological space can be made into a Kolmogorov space by identifying points with the same closure (see also [12]). The purpose is to generalize the Stone result, using Mizar System. It is shown here that: (1) *in every topological space X there exists a maximal Kolmogorov subspace X_0 of X*, and (2) *every maximal Kolmogorov subspace X_0 of X is a continuous retract of X*. Moreover, *if $r : X \to X_0$ is a continuous retraction of X onto a maximal Kolmogorov subspace X_0 of X, then $r^{-1}(x) = \mathrm{MaxADSet}(x)$ for any point x of X belonging to X_0, where $\mathrm{MaxADSet}(x)$ is a unique maximal anti-discrete subset of X containing x* (see [7] for the precise definition of the set $\mathrm{MaxADSet}(x)$). The retraction r from the last theorem is defined uniquely, and it is denoted in the text by "Stone-retraction". It has the following two remarkable properties: r is open, i.e., sends open sets in X to open sets in X_0, and r is closed, i.e., sends closed sets in X to closed sets in X_0.

These results may be obtained by the methods described by R.H. Warren in [16].

MML Identifier: `TSP_2`.

WWW: `http://mizar.org/JFM/Vol6/tsp_2.html`

The articles [13], [4], [15], [17], [2], [3], [10], [18], [14], [6], [7], and [8] provide the notation and terminology for this paper.

1. MAXIMAL T_0-SUBSETS

Let X be a non empty topological space and let A be a subset of X. Let us observe that A is T_0 if and only if:

(Def. 1) For all points a, b of X such that $a \in A$ and $b \in A$ holds if $a \neq b$, then $\mathrm{MaxADSet}(a)$ misses $\mathrm{MaxADSet}(b)$.

Let X be a non empty topological space and let A be a subset of X. Let us observe that A is T_0 if and only if:

(Def. 2) For every point a of X such that $a \in A$ holds $A \cap \mathrm{MaxADSet}(a) = \{a\}$.

[1]Presented at Mizar Conference: Mathematics in Mizar (Białystok, September 12–14, 1994).

Let X be a non empty topological space and let A be a subset of X. Let us observe that A is T_0 if and only if:

(Def. 3) For every point a of X such that $a \in A$ there exists a subset D of X such that D is maximal anti-discrete and $A \cap D = \{a\}$.

Let Y be a topological structure and let I_1 be a subset of Y. We say that I_1 is maximal T_0 if and only if:

(Def. 4) I_1 is T_0 and for every subset D of Y such that D is T_0 and $I_1 \subseteq D$ holds $I_1 = D$.

We now state the proposition

(1) Let Y_0, Y_1 be topological structures, D_0 be a subset of Y_0, and D_1 be a subset of Y_1. Suppose the topological structure of Y_0 = the topological structure of Y_1 and $D_0 = D_1$. If D_0 is maximal T_0, then D_1 is maximal T_0.

Let X be a non empty topological space and let M be a subset of X. Let us observe that M is maximal T_0 if and only if:

(Def. 5) M is T_0 and $\mathrm{MaxADSet}(M) =$ the carrier of X.

In the sequel X denotes a non empty topological space.
We now state several propositions:

(2) For every subset M of X such that M is maximal T_0 holds M is dense.

(3) For every subset A of X such that A is open and closed holds if A is maximal T_0, then A is not proper.

(4) For every empty subset A of X holds A is not maximal T_0.

(5) Let M be a subset of X. Suppose M is maximal T_0. Let A be a subset of X. If A is closed, then $A = \mathrm{MaxADSet}(M \cap A)$.

(6) Let M be a subset of X. Suppose M is maximal T_0. Let A be a subset of X. If A is open, then $A = \mathrm{MaxADSet}(M \cap A)$.

(7) For every subset M of X such that M is maximal T_0 and for every subset A of X holds $\overline{A} = \mathrm{MaxADSet}(M \cap \overline{A})$.

(8) For every subset M of X such that M is maximal T_0 and for every subset A of X holds $\mathrm{Int} A = \mathrm{MaxADSet}(M \cap \mathrm{Int} A)$.

Let X be a non empty topological space and let M be a subset of X. Let us observe that M is maximal T_0 if and only if:

(Def. 6) For every point x of X there exists a point a of X such that $a \in M$ and $M \cap \mathrm{MaxADSet}(x) = \{a\}$.

One can prove the following two propositions:

(9) For every subset A of X such that A is T_0 there exists a subset M of X such that $A \subseteq M$ and M is maximal T_0.

(10) There exists a subset of X which is maximal T_0.

2. Maximal Kolmogorov Subspaces

Let Y be a non empty topological structure and let I_1 be a subspace of Y. We say that I_1 is maximal T_0 if and only if:

(Def. 7) For every subset A of Y such that $A =$ the carrier of I_1 holds A is maximal T_0.

We now state the proposition

(11) Let Y be a non empty topological structure, Y_0 be a subspace of Y, and A be a subset of Y. Suppose $A =$ the carrier of Y_0. Then A is maximal T_0 if and only if Y_0 is maximal T_0.

Let Y be a non empty topological structure. Observe that every non empty subspace of Y which is maximal T_0 is also T_0 and every non empty subspace of Y which is non T_0 is also non maximal T_0.

Let X be a non empty topological space and let X_0 be a non empty subspace of X. Let us observe that X_0 is maximal T_0 if and only if the conditions (Def. 8) are satisfied.

(Def. 8)(i) X_0 is T_0, and

(ii) for every T_0 non empty subspace Y_0 of X such that X_0 is a subspace of Y_0 holds the topological structure of $X_0 =$ the topological structure of Y_0.

In the sequel X is a non empty topological space.

The following proposition is true

(12) Let A_0 be a non empty subset of X. Suppose A_0 is maximal T_0. Then there exists a strict non empty subspace X_0 of X such that X_0 is maximal T_0 and $A_0 =$ the carrier of X_0.

Let X be a non empty topological space. One can verify the following observations:

* every subspace of X which is maximal T_0 is also dense,
* every subspace of X which is non dense is also non maximal T_0,
* every subspace of X which is open and maximal T_0 is also non proper,
* every subspace of X which is open and proper is also non maximal T_0,
* every subspace of X which is proper and maximal T_0 is also non open,
* every subspace of X which is closed and maximal T_0 is also non proper,
* every subspace of X which is closed and proper is also non maximal T_0, and
* every subspace of X which is proper and maximal T_0 is also non closed.

Next we state the proposition

(13) Let Y_0 be a T_0 non empty subspace of X. Then there exists a strict subspace X_0 of X such that Y_0 is a subspace of X_0 and X_0 is maximal T_0.

Let X be a non empty topological space. One can check that there exists a subspace of X which is maximal T_0, strict, and non empty.

Let X be a non empty topological space. A maximal Kolmogorov subspace of X is a maximal T_0 subspace of X.

The following four propositions are true:

(14) Let X_0 be a maximal Kolmogorov subspace of X, G be a subset of X, and G_0 be a subset of X_0. Suppose $G_0 = G$. Then G_0 is open if and only if the following conditions are satisfied:

(i) $\mathrm{MaxADSet}(G)$ is open, and

(ii) $G_0 = \mathrm{MaxADSet}(G) \cap$ the carrier of X_0.

(15) Let X_0 be a maximal Kolmogorov subspace of X and G be a subset of X. Then G is open if and only if the following conditions are satisfied:

(i) $G = \mathrm{MaxADSet}(G)$, and

(ii) there exists a subset G_0 of X_0 such that G_0 is open and $G_0 = G \cap$ the carrier of X_0.

(16) Let X_0 be a maximal Kolmogorov subspace of X, F be a subset of X, and F_0 be a subset of X_0. Suppose $F_0 = F$. Then F_0 is closed if and only if the following conditions are satisfied:

(i) $\mathrm{MaxADSet}(F)$ is closed, and

(ii) $F_0 = \mathrm{MaxADSet}(F) \cap$ the carrier of X_0.

(17) Let X_0 be a maximal Kolmogorov subspace of X and F be a subset of X. Then F is closed if and only if the following conditions are satisfied:

(i) $F = \mathrm{MaxADSet}(F)$, and

(ii) there exists a subset F_0 of X_0 such that F_0 is closed and $F_0 = F \cap$ the carrier of X_0.

3. Stone Retraction Mapping Theorems

In the sequel X is a non empty topological space and X_0 is a non empty maximal Kolmogorov subspace of X.

Next we state several propositions:

(18) Let r be a map from X into X_0 and M be a subset of X. Suppose $M =$ the carrier of X_0. Suppose that for every point a of X holds $M \cap \mathrm{MaxADSet}(a) = \{r(a)\}$. Then r is a continuous map from X into X_0.

(19) Let r be a map from X into X_0. Suppose that for every point a of X holds $r(a) \in \mathrm{MaxADSet}(a)$. Then r is a continuous map from X into X_0.

(20) Let r be a continuous map from X into X_0 and M be a subset of X. Suppose $M =$ the carrier of X_0. If for every point a of X holds $M \cap \mathrm{MaxADSet}(a) = \{r(a)\}$, then r is a retraction.

(21) For every continuous map r from X into X_0 such that for every point a of X holds $r(a) \in \mathrm{MaxADSet}(a)$ holds r is a retraction.

(22) There exists a continuous map from X into X_0 which is a retraction.

(23) X_0 is a retract of X.

Let X be a non empty topological space and let X_0 be a non empty maximal Kolmogorov subspace of X. Stone-retraction of X onto X_0 is a continuous map from X into X_0 and is defined by:

(Def. 9) Stone-retraction of X onto X_0 is a retraction.

One can prove the following propositions:

(24) Let a be a point of X and b be a point of X_0. If $a = b$, then (Stone-retraction of X onto $X_0)^{-1}(\overline{\{b\}}) = \overline{\{a\}}$.

(25) For every point a of X and for every point b of X_0 such that $a = b$ holds (Stone-retraction of X onto $X_0)^{-1}(\{b\}) = \mathrm{MaxADSet}(a)$.

(26) For every subset E of X and for every subset F of X_0 such that $F = E$ holds (Stone-retraction of X onto $X_0)^{-1}(F) = \mathrm{MaxADSet}(E)$.

Let X be a non empty topological space and let X_0 be a non empty maximal Kolmogorov subspace of X. Then Stone-retraction of X onto X_0 is a continuous map from X into X_0 and it can be characterized by the condition:

(Def. 10) Let M be a subset of X. Suppose $M =$ the carrier of X_0. Let a be a point of X. Then $M \cap \mathrm{MaxADSet}(a) = \{(\text{Stone-retraction of } X \text{ onto } X_0)(a)\}$.

Let X be a non empty topological space and let X_0 be a non empty maximal Kolmogorov subspace of X. Then Stone-retraction of X onto X_0 is a continuous map from X into X_0 and it can be characterized by the condition:

(Def. 11) For every point a of X holds (Stone-retraction of X onto X_0)$(a) \in \mathrm{MaxADSet}(a)$.

We now state two propositions:

(27) For every point a of X holds (Stone-retraction of X onto X_0)$^{-1}(\{$(Stone-retraction of X onto X_0)$(a)\}) = \mathrm{MaxADSet}(a)$.

(28) For every point a of X holds (Stone-retraction of X onto X_0)$^\circ\{a\} =$ (Stone-retraction of X onto X_0)$^\circ \mathrm{MaxADSet}(a)$.

Let X be a non empty topological space and let X_0 be a non empty maximal Kolmogorov subspace of X. Then Stone-retraction of X onto X_0 is a continuous map from X into X_0 and it can be characterized by the condition:

(Def. 12) Let M be a subset of X. Suppose $M =$ the carrier of X_0. Let A be a subset of X. Then $M \cap \mathrm{MaxADSet}(A) =$ (Stone-retraction of X onto X_0)$^\circ A$.

The following propositions are true:

(29) For every subset A of X holds (Stone-retraction of X onto X_0)$^{-1}$((Stone-retraction of X onto X_0)$^\circ A) = \mathrm{MaxADSet}(A)$.

(30) For every subset A of X holds (Stone-retraction of X onto X_0)$^\circ A =$ (Stone-retraction of X onto X_0)$^\circ \mathrm{MaxADSet}(A)$.

(31) Let A, B be subsets of X. Then (Stone-retraction of X onto X_0)$^\circ(A \cup B) =$ (Stone-retraction of X onto X_0)$^\circ A \cup$ (Stone-retraction of X onto X_0)$^\circ B$.

(32) Let A, B be subsets of X. Suppose A is open or B is open. Then (Stone-retraction of X onto X_0)$^\circ(A \cap B) =$ (Stone-retraction of X onto X_0)$^\circ A \cap$ (Stone-retraction of X onto X_0)$^\circ B$.

(33) Let A, B be subsets of X. Suppose A is closed or B is closed. Then (Stone-retraction of X onto X_0)$^\circ(A \cap B) =$ (Stone-retraction of X onto X_0)$^\circ A \cap$ (Stone-retraction of X onto X_0)$^\circ B$.

(34) For every subset A of X such that A is open holds (Stone-retraction of X onto X_0)$^\circ A$ is open.

(35) For every subset A of X such that A is closed holds (Stone-retraction of X onto X_0)$^\circ A$ is closed.

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Received July 26, 1994

Published January 2, 2004