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On Kolmogorov Topological Spaces[1]

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Summary. Let X be a topological space. X is said to be T_0-*space* (or *Kolmogorov space*) provided for every pair of distinct points $x, y \in X$ there exists an open subset of X containing exactly one of these points; equivalently, for every pair of distinct points $x, y \in X$ there exists a closed subset of X containing exactly one of these points (see [1], [6], [2]).

The purpose is to list some of the standard facts on Kolmogorov spaces, using Mizar formalism. As a sample we formulate the following characteristics of such spaces: *X is a Kolmogorov space iff for every pair of distinct points $x, y \in X$ the closures $\overline{\{x\}}$ and $\overline{\{y\}}$ are distinct*.

There is also reviewed analogous facts on Kolmogorov subspaces of topological spaces. In the presented approach T_0-subsets are introduced and some of their properties developed.

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The articles [8], [10], [7], [9], [3], [4], [11], and [5] provide the notation and terminology for this paper.

1. Subspaces

Let Y be a topological structure. We see that the subspace of Y is a topological structure and it can be characterized by the following (equivalent) condition:

(Def. 1)(i) The carrier of it $\subseteq$ the carrier of Y, and

(ii) for every subset G_0 of it holds G_0 is open iff there exists a subset G of Y such that G is open and $G_0 = G \cap$ the carrier of it.

Next we state the proposition

(2)[1] Let Y be a topological structure, Y_0 be a subspace of Y, and G be a subset of Y. Suppose G is open. Then there exists a subset G_0 of Y_0 such that G_0 is open and $G_0 = G \cap$ the carrier of Y_0.

Let Y be a topological structure. We see that the subspace of Y is a topological structure and it can be characterized by the following (equivalent) condition:

(Def. 2)(i) The carrier of it $\subseteq$ the carrier of Y, and

(ii) for every subset F_0 of it holds F_0 is closed iff there exists a subset F of Y such that F is closed and $F_0 = F \cap$ the carrier of it.

[1]Presented at Mizar Conference: Mathematics in Mizar (Białystok, September 12–14, 1994).
[1] The proposition (1) has been removed.

Next we state the proposition

(4)[2] Let Y be a topological structure, Y_0 be a subspace of Y, and F be a subset of Y. Suppose F is closed. Then there exists a subset F_0 of Y_0 such that F_0 is closed and $F_0 = F \cap$ the carrier of Y_0.

2. Kolmogorov Spaces

Let T be a topological structure. Let us observe that T is discernible if and only if the conditions (Def. 3) are satisfied.

(Def. 3)(i) T is empty, or

(ii) for all points x, y of T such that $x \neq y$ holds there exists a subset V of T such that V is open and $x \in V$ and $y \notin V$ or there exists a subset W of T such that W is open and $x \notin W$ and $y \in W$.

We introduce T is T_0 as a synonym of T is discernible.

Let Y be a topological structure. Let us observe that Y is T_0 if and only if the conditions (Def. 4) are satisfied.

(Def. 4)(i) Y is empty, or

(ii) for all points x, y of Y such that $x \neq y$ holds there exists a subset E of Y such that E is closed and $x \in E$ and $y \notin E$ or there exists a subset F of Y such that F is closed and $x \notin F$ and $y \in F$.

Let us note that every non empty topological structure which is trivial is also T_0 and every non empty topological structure which is non T_0 is also non trivial.

Let us mention that there exists a topological space which is strict, T_0, and non empty and there exists a topological space which is strict, non T_0, and non empty.

One can check the following observations:

* every non empty topological space which is discrete is also T_0,
* every non empty topological space which is non T_0 is also non discrete,
* every non empty topological space which is anti-discrete and non trivial is also non T_0,
* every non empty topological space which is anti-discrete and T_0 is also trivial, and
* every non empty topological space which is T_0 and non trivial is also non anti-discrete.

Let X be a non empty topological space. Let us observe that X is T_0 if and only if:

(Def. 5) For all points x, y of X such that $x \neq y$ holds $\overline{\{x\}} \neq \overline{\{y\}}$.

Let X be a non empty topological space. Let us observe that X is T_0 if and only if:

(Def. 6) For all points x, y of X such that $x \neq y$ holds $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$.

Let X be a non empty topological space. Let us observe that X is T_0 if and only if:

(Def. 7) For all points x, y of X such that $x \neq y$ and $x \in \overline{\{y\}}$ holds $\overline{\{y\}} \not\subseteq \overline{\{x\}}$.

One can verify the following observations:

* every non empty topological space which is almost discrete and T_0 is also discrete,
* every non empty topological space which is almost discrete and non discrete is also non T_0, and
* every non empty topological space which is non discrete and T_0 is also non almost discrete.

A Kolmogorov space is a T_0 non empty topological space. A non-Kolmogorov space is a non T_0 non empty topological space.

Let us mention that there exists a Kolmogorov space which is non trivial and strict and there exists a non-Kolmogorov space which is non trivial and strict.

[2] The proposition (3) has been removed.

3. T_0-SUBSETS

Let Y be a topological structure and let I_1 be a subset of Y. We say that I_1 is T_0 if and only if the condition (Def. 8) is satisfied.

(Def. 8) Let x, y be points of Y. Suppose $x \in I_1$ and $y \in I_1$ and $x \neq y$. Then there exists a subset V of Y such that V is open and $x \in V$ and $y \notin V$ or there exists a subset W of Y such that W is open and $x \notin W$ and $y \in W$.

Let Y be a non empty topological structure and let A be a subset of Y. Let us observe that A is T_0 if and only if the condition (Def. 9) is satisfied.

(Def. 9) Let x, y be points of Y. Suppose $x \in A$ and $y \in A$ and $x \neq y$. Then

(i) there exists a subset E of Y such that E is closed and $x \in E$ and $y \notin E$, or

(ii) there exists a subset F of Y such that F is closed and $x \notin F$ and $y \in F$.

Next we state two propositions:

(5) Let Y_0, Y_1 be topological structures, D_0 be a subset of Y_0, and D_1 be a subset of Y_1. Suppose the topological structure of Y_0 = the topological structure of Y_1 and $D_0 = D_1$. If D_0 is T_0, then D_1 is T_0.

(6) Let Y be a non empty topological structure and A be a subset of Y. Suppose A = the carrier of Y. Then A is T_0 if and only if Y is T_0.

In the sequel Y is a non empty topological structure.
One can prove the following propositions:

(7) For all subsets A, B of Y such that $B \subseteq A$ holds if A is T_0, then B is T_0.

(8) For all subsets A, B of Y such that A is T_0 or B is T_0 holds $A \cap B$ is T_0.

(9) Let A, B be subsets of Y. Suppose A is open or B is open. If A is T_0 and B is T_0, then $A \cup B$ is T_0.

(10) Let A, B be subsets of Y. Suppose A is closed or B is closed. If A is T_0 and B is T_0, then $A \cup B$ is T_0.

(11) For every subset A of Y such that A is discrete holds A is T_0.

(12) For every non empty subset A of Y such that A is anti-discrete and A is not trivial holds A is not T_0.

Let X be a non empty topological space and let A be a subset of X. Let us observe that A is T_0 if and only if:

(Def. 10) For all points x, y of X such that $x \in A$ and $y \in A$ and $x \neq y$ holds $\overline{\{x\}} \neq \overline{\{y\}}$.

Let X be a non empty topological space and let A be a subset of X. Let us observe that A is T_0 if and only if:

(Def. 11) For all points x, y of X such that $x \in A$ and $y \in A$ and $x \neq y$ holds $x \notin \overline{\{y\}}$ or $y \notin \overline{\{x\}}$.

Let X be a non empty topological space and let A be a subset of X. Let us observe that A is T_0 if and only if:

(Def. 12) For all points x, y of X such that $x \in A$ and $y \in A$ and $x \neq y$ holds if $x \in \overline{\{y\}}$, then $\overline{\{y\}} \not\subseteq \overline{\{x\}}$.

In the sequel X denotes a non empty topological space.
Next we state two propositions:

(13) Every empty subset of X is T_0.

(14) For every point x of X holds $\{x\}$ is T_0.

4. Kolmogorov Subspaces

Let Y be a non empty topological structure. One can verify that there exists a subspace of Y which is strict, T_0, and non empty.

Let Y be a topological structure and let Y_0 be a subspace of Y. Let us observe that Y_0 is T_0 if and only if the conditions (Def. 13) are satisfied.

(Def. 13)(i) Y_0 is empty, or

(ii) for all points x, y of Y such that x is a point of Y_0 and y is a point of Y_0 and $x \neq y$ holds there exists a subset V of Y such that V is open and $x \in V$ and $y \notin V$ or there exists a subset W of Y such that W is open and $x \notin W$ and $y \in W$.

Let Y be a topological structure and let Y_0 be a subspace of Y. Let us observe that Y_0 is T_0 if and only if the conditions (Def. 14) are satisfied.

(Def. 14)(i) Y_0 is empty, or

(ii) for all points x, y of Y such that x is a point of Y_0 and y is a point of Y_0 and $x \neq y$ holds there exists a subset E of Y such that E is closed and $x \in E$ and $y \notin E$ or there exists a subset F of Y such that F is closed and $x \notin F$ and $y \in F$.

In the sequel Y denotes a non empty topological structure.

Next we state two propositions:

(15) Let Y_0 be a non empty subspace of Y and A be a subset of Y. Suppose $A =$ the carrier of Y_0. Then A is T_0 if and only if Y_0 is T_0.

(16) Let Y_0 be a non empty subspace of Y and Y_1 be a T_0 non empty subspace of Y. If Y_0 is a subspace of Y_1, then Y_0 is T_0.

In the sequel X is a non empty topological space.

We now state three propositions:

(17) Let X_1 be a T_0 non empty subspace of X and X_2 be a non empty subspace of X. If X_1 meets X_2, then $X_1 \cap X_2$ is T_0.

(18) For all T_0 non empty subspaces X_1, X_2 of X such that X_1 is open or X_2 is open holds $X_1 \cup X_2$ is T_0.

(19) For all T_0 non empty subspaces X_1, X_2 of X such that X_1 is closed or X_2 is closed holds $X_1 \cup X_2$ is T_0.

Let X be a non empty topological space. A Kolmogorov subspace of X is a T_0 non empty subspace of X.

The following proposition is true

(20) Let X be a non empty topological space and A_0 be a non empty subset of X. Suppose A_0 is T_0. Then there exists a strict Kolmogorov subspace X_0 of X such that $A_0 =$ the carrier of X_0.

Let X be a non trivial non empty topological space. Observe that there exists a Kolmogorov subspace of X which is proper and strict.

Let X be a Kolmogorov space. One can verify that every non empty subspace of X is T_0.

Let X be a non-Kolmogorov space. Note that every non empty subspace of X which is non proper is also non T_0 and every non empty subspace of X which is T_0 is also proper.

Let X be a non-Kolmogorov space. Observe that there exists a subspace of X which is strict and non T_0.

Let X be a non-Kolmogorov space. A non-Kolmogorov subspace of X is a non T_0 subspace of X.

We now state the proposition

(21) Let X be a non empty non-Kolmogorov space and A_0 be a subset of X. Suppose A_0 is not T_0. Then there exists a strict non-Kolmogorov subspace X_0 of X such that $A_0 =$ the carrier of X_0.

REFERENCES

[1] P. Alexandroff and H. H. Hopf. *Topologie I.* Springer-Verlag, Berlin, 1935.

[2] Ryszard Engelking. *General Topology*, volume 60 of *Monografie Matematyczne*. PWN - Polish Scientific Publishers, Warsaw, 1977.

[3] Zbigniew Karno. Separated and weakly separated subspaces of topological spaces. *Journal of Formalized Mathematics*, 4, 1992. http://mizar.org/JFM/Vol4/tsep_1.html.

[4] Zbigniew Karno. Maximal discrete subspaces of almost discrete topological spaces. *Journal of Formalized Mathematics*, 5, 1993. http://mizar.org/JFM/Vol5/tex_2.html.

[5] Zbigniew Karno. Maximal anti-discrete subspaces of topological spaces. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/tex_4.html.

[6] Kazimierz Kuratowski. *Topology*, volume I. PWN - Polish Scientific Publishers, Academic Press, Warsaw, New York and London, 1966.

[7] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/pre_topc.html.

[8] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[9] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/borsuk_1.html.

[10] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/subset_1.html.

[11] Mariusz Żynel and Adam Guzowski. T_0 topological spaces. *Journal of Formalized Mathematics*, 6, 1994. http://mizar.org/JFM/Vol6/t_0topsp.html.

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