JOURNAL OF FORMALIZED MATHEMATICS
Volume 5, Released 1993, Published 2003
Inst. of Computer Science, Univ. of Białystok

Joining of Decorated Trees

Grzegorz Bancerek
Polish Academy of Sciences
Institute of Mathematics
Warsaw

Summary. This is the continuation of the sequence of articles on trees (see [2], [4], [5]). The main goal is to introduce joining operations on decorated trees corresponding with operations introduced in [5]. We will also introduce the operation of substitution. In the last section we dealt with trees decorated by Cartesian product, i.e. we showed some lemmas on joining operations applied to such trees.

MML Identifier: TREES_4.

WWW: http://mizar.org/JFM/Vol5/trees_4.html

The articles [13], [9], [15], [14], [1], [16], [8], [10], [12], [11], [7], [6], [2], [4], [3], and [5] provide the notation and terminology for this paper.

1. Joining of Decorated Tree

Let T be a decorated tree. A node of T is an element of $\operatorname{dom} T$.

We adopt the following convention: x, y are sets, i, j, n are natural numbers, and p, q are finite sequences.

Let T_1, T_2 be decorated trees. Let us observe that $T_1 = T_2$ if and only if:

(Def. 1) $\operatorname{dom} T_1 = \operatorname{dom} T_2$ and for every node p of T_1 holds $T_1(p) = T_2(p)$.

One can prove the following propositions:

(1) For all natural numbers i, j such that the elementary tree of $i \subseteq$ the elementary tree of j holds $i \leq j$.

(2) For all natural numbers i, j such that the elementary tree of $i =$ the elementary tree of j holds $i = j$.

Let us consider x. The root tree of x is a decorated tree and is defined as follows:

(Def. 2) The root tree of $x =$ (the elementary tree of 0) $\longmapsto x$.

Let D be a non empty set and let d be an element of D. Then the root tree of d is an element of $\operatorname{FinTrees}(D)$.

One can prove the following four propositions:

(3) dom(the root tree of x) $=$ the elementary tree of 0 and (the root tree of x)$(\emptyset) = x$.

(4) If the root tree of $x =$ the root tree of y, then $x = y$.

(5) For every decorated tree T such that $\operatorname{dom} T =$ the elementary tree of 0 holds $T =$ the root tree of $T(\emptyset)$.

(6) The root tree of $x = \{\langle \emptyset, x \rangle\}$.

Let us consider x and let p be a finite sequence. The flat tree of x and p is a decorated tree and is defined by the conditions (Def. 3).

(Def. 3)(i) dom(the flat tree of x and p) = the elementary tree of $\operatorname{len} p$,

(ii) (the flat tree of x and p)$(\emptyset) = x$, and

(iii) for every n such that $n < \operatorname{len} p$ holds (the flat tree of x and p)$(\langle n \rangle) = p(n+1)$.

Next we state three propositions:

(7) If the flat tree of x and $p =$ the flat tree of y and q, then $x = y$ and $p = q$.

(8) If $j < i$, then (the elementary tree of i)$\restriction \langle j \rangle =$ the elementary tree of 0.

(9) If $i < \operatorname{len} p$, then (the flat tree of x and p)$\restriction \langle i \rangle =$ the root tree of $p(i+1)$.

Let us consider x, p. Let us assume that p is decorated tree yielding. The functor x-tree(p) yields a decorated tree and is defined by the conditions (Def. 4).

(Def. 4)(i) There exists a decorated tree yielding finite sequence q such that $p = q$ and $\operatorname{dom}(x\text{-tree}(p)) = \overbrace{\operatorname{dom}_{\kappa} q(\kappa)}$,

(ii) $(x\text{-tree}(p))(\emptyset) = x$, and

(iii) for every n such that $n < \operatorname{len} p$ holds $(x\text{-tree}(p))\restriction \langle n \rangle = p(n+1)$.

Let us consider x and let T be a decorated tree. The functor x-tree(T) yields a decorated tree and is defined by:

(Def. 5) $x\text{-tree}(T) = x\text{-tree}(\langle T \rangle)$.

Let us consider x and let T_1, T_2 be decorated trees. The functor x-tree(T_1, T_2) yields a decorated tree and is defined by:

(Def. 6) $x\text{-tree}(T_1, T_2) = x\text{-tree}(\langle T_1, T_2 \rangle)$.

One can prove the following propositions:

(10) For every decorated tree yielding finite sequence p holds $\operatorname{dom}(x\text{-tree}(p)) = \overbrace{\operatorname{dom}_{\kappa} p(\kappa)}$.

(11) Let p be a decorated tree yielding finite sequence. Then $y \in \operatorname{dom}(x\text{-tree}(p))$ if and only if one of the following conditions is satisfied:

(i) $y = \emptyset$, or

(ii) there exists a natural number i and there exists a decorated tree T and there exists a node q of T such that $i < \operatorname{len} p$ and $T = p(i+1)$ and $y = \langle i \rangle \frown q$.

(12) Let p be a decorated tree yielding finite sequence, i be a natural number, T be a decorated tree, and q be a node of T. If $i < \operatorname{len} p$ and $T = p(i+1)$, then $(x\text{-tree}(p))(\langle i \rangle \frown q) = T(q)$.

(13) For every decorated tree T holds $\operatorname{dom}(x\text{-tree}(T)) = \overbrace{\operatorname{dom} T}$.

(14) For all decorated trees T_1, T_2 holds $\operatorname{dom}(x\text{-tree}(T_1, T_2)) = \overbrace{\operatorname{dom} T_1, \operatorname{dom} T_2}$.

(15) For all decorated tree yielding finite sequences p, q such that $x\text{-tree}(p) = y\text{-tree}(q)$ holds $x = y$ and $p = q$.

(16) If the root tree of x = the flat tree of y and p, then $x = y$ and $p = \emptyset$.

(17) If the root tree of x = y-tree(p) and p is decorated tree yielding, then $x = y$ and $p = \emptyset$.

(18) Suppose the flat tree of x and p = y-tree(q) and q is decorated tree yielding. Then $x = y$ and $\mathrm{len}\, p = \mathrm{len}\, q$ and for every i such that $i \in \mathrm{dom}\, p$ holds $q(i)$ = the root tree of $p(i)$.

(19) Let p be a decorated tree yielding finite sequence, n be a natural number, and q be a finite sequence. If $\langle n \rangle \frown q \in \mathrm{dom}(x\text{-tree}(p))$, then $(x\text{-tree}(p))(\langle n \rangle \frown q) = p(n+1)(q)$.

(20) The flat tree of x and $\emptyset$ = the root tree of x and x-tree($\emptyset$) = the root tree of x.

(21) The flat tree of x and $\langle y \rangle$ = ((the elementary tree of 1) $\longmapsto x$) with-replacement($\langle 0 \rangle$, the root tree of y).

(22) For every decorated tree T holds x-tree($\langle T \rangle$) = ((the elementary tree of 1) $\longmapsto x$) with-replacement($\langle 0 \rangle, T$).

Let D be a non empty set, let d be an element of D, and let p be a finite sequence of elements of D. Then the flat tree of d and p is a tree decorated with elements of D.

Let D be a non empty set, let F be a non empty set of trees decorated with elements of D, let d be an element of D, and let p be a finite sequence of elements of F. Then d-tree(p) is a tree decorated with elements of D.

Let D be a non empty set, let d be an element of D, and let T be a tree decorated with elements of D. Then d-tree(T) is a tree decorated with elements of D.

Let D be a non empty set, let d be an element of D, and let T_1, T_2 be trees decorated with elements of D. Then d-tree(T_1, T_2) is a tree decorated with elements of D.

Let D be a non empty set and let p be a finite sequence of elements of FinTrees(D). Then $\mathrm{dom}_\kappa\, p(\kappa)$ is a finite sequence of elements of FinTrees.

Let D be a non empty set, let d be an element of D, and let p be a finite sequence of elements of FinTrees(D). Then d-tree(p) is an element of FinTrees(D).

Let D be a non empty set and let x be a subset of D. We see that the finite sequence of elements of x is a finite sequence of elements of D.

Let D be a non empty constituted of decorated trees set and let X be a subset of D. Note that every finite sequence of elements of X is decorated tree yielding.

2. Expanding of Decorated Tree by Substitution

The scheme *ExpandTree* deals with a tree $\mathcal{A}$, a tree $\mathcal{B}$, and a unary predicate $\mathcal{P}$, and states that:

> There exists a tree T such that for every p holds $p \in T$ if and only if one of the following conditions is satisfied:
> (i) $p \in \mathcal{A}$, or
> (ii) there exists an element q of $\mathcal{A}$ and there exists an element r of $\mathcal{B}$ such that $\mathcal{P}[q]$ and $p = q \frown r$

for all values of the parameters.

Let T, T' be decorated trees and let x be a set. The functor $T_{x \leftarrow T'}$ yielding a decorated tree is defined by the conditions (Def. 7).

(Def. 7)(i) For every p holds $p \in \mathrm{dom}(T_{x \leftarrow T'})$ iff $p \in \mathrm{dom}\, T$ or there exists a node q of T and there exists a node r of T' such that $q \in \mathrm{Leaves}(\mathrm{dom}\, T)$ and $T(q) = x$ and $p = q \frown r$,

(ii) for every node p of T such that $p \notin \mathrm{Leaves}(\mathrm{dom}\, T)$ or $T(p) \neq x$ holds $T_{x \leftarrow T'}(p) = T(p)$, and

(iii) for every node p of T and for every node q of T' such that $p \in \mathrm{Leaves}(\mathrm{dom}\, T)$ and $T(p) = x$ holds $T_{x \leftarrow T'}(p \frown q) = T'(q)$.

Let D be a non empty set, let T, T' be trees decorated with elements of D, and let x be a set. Then $T_{x \leftarrow T'}$ is a tree decorated with elements of D.

In the sequel T, T' are decorated trees and x, y are sets.

Next we state the proposition

(23) If $x \notin \operatorname{rng} T$ or $x \notin \mathrm{Leaves}(T)$, then $T_{x \leftarrow T'} = T$.

3. Double Decorated Trees

For simplicity, we adopt the following convention: D_1, D_2 denote non empty sets, T denotes a tree decorated with elements of D_1 and D_2, d_1 denotes an element of D_1, d_2 denotes an element of D_2, F denotes a non empty set of trees decorated with elements of D_1 and D_2, F_1 denotes a non empty set of trees decorated with elements of D_1, and F_2 denotes a non empty set of trees decorated with elements of D_2.

We now state several propositions:

(24) For all D_1, D_2, T holds $\mathrm{dom}(T_{\mathbf{1}}) = \mathrm{dom}\, T$ and $\mathrm{dom}(T_{\mathbf{2}}) = \mathrm{dom}\, T$.

(25) (the root tree of $\langle d_1, d_2 \rangle)_{\mathbf{1}}$ = the root tree of d_1 and (the root tree of $\langle d_1, d_2 \rangle)_{\mathbf{2}}$ = the root tree of d_2.

(26) $\langle$the root tree of x, the root tree of $y\rangle$ = the root tree of $\langle x, y \rangle$.

(27) Let given D_1, D_2, d_1, d_2, F, F_1, p be a finite sequence of elements of F, and p_1 be a finite sequence of elements of F_1. Suppose $\mathrm{dom}\, p_1 = \mathrm{dom}\, p$ and for every i such that $i \in \mathrm{dom}\, p$ and for every T such that $T = p(i)$ holds $p_1(i) = T_{\mathbf{1}}$. Then $(\langle d_1, d_2 \rangle\text{-tree}(p))_{\mathbf{1}} = d_1\text{-tree}(p_1)$.

(28) Let given D_1, D_2, d_1, d_2, F, F_2, p be a finite sequence of elements of F, and p_2 be a finite sequence of elements of F_2. Suppose $\mathrm{dom}\, p_2 = \mathrm{dom}\, p$ and for every i such that $i \in \mathrm{dom}\, p$ and for every T such that $T = p(i)$ holds $p_2(i) = T_{\mathbf{2}}$. Then $(\langle d_1, d_2 \rangle\text{-tree}(p))_{\mathbf{2}} = d_2\text{-tree}(p_2)$.

(29) Let given D_1, D_2, d_1, d_2, F and p be a finite sequence of elements of F. Then there exists a finite sequence p_1 of elements of $\mathrm{Trees}(D_1)$ such that $\mathrm{dom}\, p_1 = \mathrm{dom}\, p$ and for every i such that $i \in \mathrm{dom}\, p$ there exists an element T of F such that $T = p(i)$ and $p_1(i) = T_{\mathbf{1}}$ and $(\langle d_1, d_2 \rangle\text{-tree}(p))_{\mathbf{1}} = d_1\text{-tree}(p_1)$.

(30) Let given D_1, D_2, d_1, d_2, F and p be a finite sequence of elements of F. Then there exists a finite sequence p_2 of elements of $\mathrm{Trees}(D_2)$ such that $\mathrm{dom}\, p_2 = \mathrm{dom}\, p$ and for every i such that $i \in \mathrm{dom}\, p$ there exists an element T of F such that $T = p(i)$ and $p_2(i) = T_{\mathbf{2}}$ and $(\langle d_1, d_2 \rangle\text{-tree}(p))_{\mathbf{2}} = d_2\text{-tree}(p_2)$.

(31) Let given D_1, D_2, d_1, d_2 and p be a finite sequence of elements of $\mathrm{FinTrees}([\!: D_1, D_2 :\!])$. Then there exists a finite sequence p_1 of elements of $\mathrm{FinTrees}(D_1)$ such that $\mathrm{dom}\, p_1 = \mathrm{dom}\, p$ and for every i such that $i \in \mathrm{dom}\, p$ there exists an element T of $\mathrm{FinTrees}([\!: D_1, D_2 :\!])$ such that $T = p(i)$ and $p_1(i) = T_{\mathbf{1}}$ and $(\langle d_1, d_2 \rangle\text{-tree}(p))_{\mathbf{1}} = d_1\text{-tree}(p_1)$.

(32) Let given D_1, D_2, d_1, d_2 and p be a finite sequence of elements of $\mathrm{FinTrees}([\!: D_1, D_2 :\!])$. Then there exists a finite sequence p_2 of elements of $\mathrm{FinTrees}(D_2)$ such that $\mathrm{dom}\, p_2 = \mathrm{dom}\, p$ and for every i such that $i \in \mathrm{dom}\, p$ there exists an element T of $\mathrm{FinTrees}([\!: D_1, D_2 :\!])$ such that $T = p(i)$ and $p_2(i) = T_{\mathbf{2}}$ and $(\langle d_1, d_2 \rangle\text{-tree}(p))_{\mathbf{2}} = d_2\text{-tree}(p_2)$.

References

[1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/nat_1.html`.

[2] Grzegorz Bancerek. Introduction to trees. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/trees_1.html`.

[3] Grzegorz Bancerek. Cartesian product of functions. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/funct_6.html`.

[4] Grzegorz Bancerek. König's Lemma. *Journal of Formalized Mathematics*, 3, 1991. `http://mizar.org/JFM/Vol3/trees_2.html`.

[5] Grzegorz Bancerek. Sets and functions of trees and joining operations of trees. *Journal of Formalized Mathematics*, 4, 1992. `http://mizar.org/JFM/Vol4/trees_3.html`.

[6] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finseq_1.html`.

[7] Czesław Byliński. Basic functions and operations on functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_3.html`.

[8] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funct_1.html`.

[9] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/zfmisc_1.html`.

[10] Agata Darmochwał. Finite sets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/finset_1.html`.

[11] Andrzej Trybulec. Binary operations applied to functions. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/funcop_1.html`.

[12] Andrzej Trybulec. Domains and their Cartesian products. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/domain_1.html`.

[13] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. `http://mizar.org/JFM/Axiomatics/tarski.html`.

[14] Andrzej Trybulec. Subsets of real numbers. *Journal of Formalized Mathematics*, Addenda, 2003. `http://mizar.org/JFM/Addenda/numbers.html`.

[15] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[16] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

Received October 8, 1993

Published January 2, 2004