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The Brouwer Fixed Point Theorem for Intervals[1]

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Summary. The aim is to prove, using Mizar System, the following simplest version of the Brouwer Fixed Point Theorem [3]. *For every continuous mapping* $f : \mathbb{I} \to \mathbb{I}$ *of the topological unit interval* $\mathbb{I}$ *there exists a point* x *such that* $f(x) = x$ (see e.g. [9], [4]).

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The articles [17], [20], [1], [19], [21], [5], [6], [10], [16], [15], [7], [14], [11], [13], [2], [8], [12], and [18] provide the notation and terminology for this paper.

1. Properties of Topological Intervals

In this paper a, b, c, d are real numbers.

One can prove the following propositions:

(1) If $a \leq c$ and $d \leq b$, then $[c,d] \subseteq [a,b]$.

(2) If $a \leq c$ and $b \leq d$ and $c \leq b$, then $[a,b] \cup [c,d] = [a,d]$.

(3) If $a \leq c$ and $b \leq d$ and $c \leq b$, then $[a,b] \cap [c,d] = [c,b]$.

(4) For every subset A of $\mathbb{R}^{\mathbf{1}}$ such that $A = [a,b]$ holds A is closed.

(5) If $a \leq b$, then $[a, b]_{\mathrm{T}}$ is a closed subspace of $\mathbb{R}^{\mathbf{1}}$.

(6) If $a \leq c$ and $d \leq b$ and $c \leq d$, then $[c, d]_{\mathrm{T}}$ is a closed subspace of $[a, b]_{\mathrm{T}}$.

(7) If $a \leq c$ and $b \leq d$ and $c \leq b$, then $[a, d]_{\mathrm{T}} = [a, b]_{\mathrm{T}} \cup [c, d]_{\mathrm{T}}$ and $[c, b]_{\mathrm{T}} = [a, b]_{\mathrm{T}} \cap [c, d]_{\mathrm{T}}$.

Let a, b be real numbers. Let us assume that $a \leq b$. The functor $a_{[a,b]_{\mathrm{T}}}$ yields a point of $[a, b]_{\mathrm{T}}$ and is defined as follows:

(Def. 1) $a_{[a,b]_{\mathrm{T}}} = a$.

The functor $b_{[a,b]_{\mathrm{T}}}$ yields a point of $[a, b]_{\mathrm{T}}$ and is defined by:

(Def. 2) $b_{[a,b]_{\mathrm{T}}} = b$.

One can prove the following two propositions:

(8) $0_{\mathbb{I}} = 0_{[0,1]_{\mathrm{T}}}$ and $1_{\mathbb{I}} = 1_{[0,1]_{\mathrm{T}}}$.

(9) If $a \leq b$ and $b \leq c$, then $a_{[a,b]_{\mathrm{T}}} = a_{[a,c]_{\mathrm{T}}}$ and $c_{[b,c]_{\mathrm{T}}} = c_{[a,c]_{\mathrm{T}}}$.

[1] This paper was done under the supervision of Z. Karno while the author was visiting the Institute of Mathematics of Warsaw University in Białystok.

2. Continuous Mappings Between Topological Intervals

Let a, b be real numbers. Let us assume that $a \leq b$. Let t_1, t_2 be points of $[a, b]_{\rm T}$. The functor $\mathrm{L}_{01}(t_1,t_2)$ yielding a map from $[0, 1]_{\rm T}$ into $[a, b]_{\rm T}$ is defined as follows:

(Def. 3) For every point s of $[0, 1]_{\rm T}$ and for all real numbers r, r_1, r_2 such that $s = r$ and $r_1 = t_1$ and $r_2 = t_2$ holds $(\mathrm{L}_{01}(t_1,t_2))(s) = (1-r) \cdot r_1 + r \cdot r_2$.

Next we state four propositions:

(10) Suppose $a \leq b$. Let t_1, t_2 be points of $[a, b]_{\rm T}$, s be a point of $[0, 1]_{\rm T}$, and r, r_1, r_2 be real numbers. If $s = r$ and $r_1 = t_1$ and $r_2 = t_2$, then $(\mathrm{L}_{01}(t_1,t_2))(s) = (r_2 - r_1) \cdot r + r_1$.

(11) If $a \leq b$, then for all points t_1, t_2 of $[a, b]_{\rm T}$ holds $\mathrm{L}_{01}(t_1,t_2)$ is a continuous map from $[0, 1]_{\rm T}$ into $[a, b]_{\rm T}$.

(12) If $a \leq b$, then for all points t_1, t_2 of $[a, b]_{\rm T}$ holds $(\mathrm{L}_{01}(t_1,t_2))(0_{[0,1]_{\rm T}}) = t_1$ and $(\mathrm{L}_{01}(t_1,t_2))(1_{[0,1]_{\rm T}}) = t_2$.

(13) $\mathrm{L}_{01}(0_{[0,1]_{\rm T}}, 1_{[0,1]_{\rm T}}) = \mathrm{id}_{[0,\,1]_{\rm T}}$.

Let a, b be real numbers. Let us assume that $a < b$. Let t_1, t_2 be points of $[0, 1]_{\rm T}$. The functor $\mathrm{P}_{01}(a,b,t_1,t_2)$ yielding a map from $[a, b]_{\rm T}$ into $[0, 1]_{\rm T}$ is defined by the condition (Def. 4).

(Def. 4) Let s be a point of $[a, b]_{\rm T}$ and r, r_1, r_2 be real numbers. If $s = r$ and $r_1 = t_1$ and $r_2 = t_2$, then $(\mathrm{P}_{01}(a,b,t_1,t_2))(s) = \frac{(b-r)\cdot r_1 + (r-a)\cdot r_2}{b-a}$.

Next we state several propositions:

(14) Suppose $a < b$. Let t_1, t_2 be points of $[0, 1]_{\rm T}$, s be a point of $[a, b]_{\rm T}$, and r, r_1, r_2 be real numbers. If $s = r$ and $r_1 = t_1$ and $r_2 = t_2$, then $(\mathrm{P}_{01}(a,b,t_1,t_2))(s) = \frac{r_2 - r_1}{b-a} \cdot r + \frac{b \cdot r_1 - a \cdot r_2}{b-a}$.

(15) If $a < b$, then for all points t_1, t_2 of $[0, 1]_{\rm T}$ holds $\mathrm{P}_{01}(a,b,t_1,t_2)$ is a continuous map from $[a, b]_{\rm T}$ into $[0, 1]_{\rm T}$.

(16) If $a < b$, then for all points t_1, t_2 of $[0, 1]_{\rm T}$ holds $(\mathrm{P}_{01}(a,b,t_1,t_2))(a_{[a,b]_{\rm T}}) = t_1$ and $(\mathrm{P}_{01}(a,b,t_1,t_2))(b_{[a,b]_{\rm T}}) = t_2$.

(17) $\mathrm{P}_{01}(0,1,0_{[0,1]_{\rm T}},1_{[0,1]_{\rm T}}) = \mathrm{id}_{[0,\,1]_{\rm T}}$.

(18) If $a < b$, then $\mathrm{id}_{[a,b]_{\rm T}} = \mathrm{L}_{01}(a_{[a,b]_{\rm T}}, b_{[a,b]_{\rm T}}) \cdot \mathrm{P}_{01}(a,b,0_{[0,1]_{\rm T}},1_{[0,1]_{\rm T}})$ and $\mathrm{id}_{[0,\,1]_{\rm T}} = \mathrm{P}_{01}(a,b,0_{[0,1]_{\rm T}},1_{[0,1]_{\rm T}}) \cdot \mathrm{L}_{01}(a_{[a,b]_{\rm T}}, b_{[a,b]_{\rm T}})$.

(19) If $a < b$, then $\mathrm{id}_{[a,b]_{\rm T}} = \mathrm{L}_{01}(b_{[a,b]_{\rm T}}, a_{[a,b]_{\rm T}}) \cdot \mathrm{P}_{01}(a,b,1_{[0,1]_{\rm T}},0_{[0,1]_{\rm T}})$ and $\mathrm{id}_{[0,\,1]_{\rm T}} = \mathrm{P}_{01}(a,b,1_{[0,1]_{\rm T}},0_{[0,1]_{\rm T}}) \cdot \mathrm{L}_{01}(b_{[a,b]_{\rm T}}, a_{[a,b]_{\rm T}})$.

(20) Suppose $a < b$. Then $\mathrm{L}_{01}(a_{[a,b]_{\rm T}}, b_{[a,b]_{\rm T}})$ is a homeomorphism and $(\mathrm{L}_{01}(a_{[a,b]_{\rm T}}, b_{[a,b]_{\rm T}}))^{-1} = \mathrm{P}_{01}(a,b,0_{[0,1]_{\rm T}},1_{[0,1]_{\rm T}})$ and $\mathrm{P}_{01}(a,b,0_{[0,1]_{\rm T}},1_{[0,1]_{\rm T}})$ is a homeomorphism and $(\mathrm{P}_{01}(a,b,0_{[0,1]_{\rm T}},1_{[0,1]_{\rm T}}))^{-1} = \mathrm{L}_{01}(a_{[a,b]_{\rm T}}, b_{[a,b]_{\rm T}})$.

(21) Suppose $a < b$. Then $\mathrm{L}_{01}(b_{[a,b]_{\rm T}}, a_{[a,b]_{\rm T}})$ is a homeomorphism and $(\mathrm{L}_{01}(b_{[a,b]_{\rm T}}, a_{[a,b]_{\rm T}}))^{-1} = \mathrm{P}_{01}(a,b,1_{[0,1]_{\rm T}},0_{[0,1]_{\rm T}})$ and $\mathrm{P}_{01}(a,b,1_{[0,1]_{\rm T}},0_{[0,1]_{\rm T}})$ is a homeomorphism and $(\mathrm{P}_{01}(a,b,1_{[0,1]_{\rm T}},0_{[0,1]_{\rm T}}))^{-1} = \mathrm{L}_{01}(b_{[a,b]_{\rm T}}, a_{[a,b]_{\rm T}})$.

3. Connectedness of Intervals and Brouwer Fixed Point Theorem for Intervals

We now state several propositions:

(22) $\mathbb{I}$ is connected.

(23) If $a \leq b$, then $[a, b]_{\mathrm{T}}$ is connected.

(24) For every continuous map f from $\mathbb{I}$ into $\mathbb{I}$ there exists a point x of $\mathbb{I}$ such that $f(x) = x$.

(25) If $a \leq b$, then for every continuous map f from $[a, b]_{\mathrm{T}}$ into $[a, b]_{\mathrm{T}}$ there exists a point x of $[a, b]_{\mathrm{T}}$ such that $f(x) = x$.

(26) Let X, Y be non empty subspaces of $\mathbb{R}^{\mathbf{1}}$ and f be a continuous map from X into Y. Given real numbers a, b such that $a \leq b$ and $[a,b] \subseteq$ the carrier of X and $[a,b] \subseteq$ the carrier of Y and $f^{\circ}[a,b] \subseteq [a,b]$. Then there exists a point x of X such that $f(x) = x$.

(27) Let X, Y be non empty subspaces of $\mathbb{R}^{\mathbf{1}}$ and f be a continuous map from X into Y. Given real numbers a, b such that $a \leq b$ and $[a,b] \subseteq$ the carrier of X and $f^{\circ}[a,b] \subseteq [a,b]$. Then there exists a point x of X such that $f(x) = x$.

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