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Compactness of the Bounded Closed Subsets of $\mathcal{E}^2_{\mathrm{T}}$

Artur Korniłowicz
University of Białystok

Summary. This paper contains theorems which describe the correspondence between topological properties of real numbers subsets introduced in [35] and introduced in [33], [14]. We also show the homeomorphism between the cartesian product of two R^1 and $\mathcal{E}^2_{\mathrm{T}}$. The compactness of the bounded closed subset of $\mathcal{E}^2_{\mathrm{T}}$ is proven.

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The articles [36], [9], [42], [43], [7], [8], [6], [16], [2], [38], [21], [39], [1], [34], [30], [11], [25], [24], [23], [22], [20], [4], [10], [12], [26], [3], [41], [35], [33], [15], [31], [32], [14], [37], [5], [17], [18], [19], [44], [28], [13], [27], [40], and [29] provide the notation and terminology for this paper.

1. Real Numbers

For simplicity, we adopt the following convention: a, b denote real numbers, r denotes a real number, i, j, n denote natural numbers, M denotes a non empty metric space, p, q, s denote points of $\mathcal{E}^2_{\mathrm{T}}$, e denotes a point of $\mathcal{E}^2$, w denotes a point of $\mathcal{E}^n$, z denotes a point of M, A, B denote subsets of $\mathcal{E}^n_{\mathrm{T}}$, P denotes a subset of $\mathcal{E}^2_{\mathrm{T}}$, and D denotes a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$.

One can prove the following propositions:

(6)[1] If $0 \leq a$ and $0 \leq b$, then $\sqrt{a+b} \leq \sqrt{a}+\sqrt{b}$.

(7) If $0 \leq a$ and $a \leq b$, then $|a| \leq |b|$.

(8) If $b \leq a$ and $a \leq 0$, then $|a| \leq |b|$.

(9) $\prod(0 \mapsto r) = 1$.

(10) $\prod(1 \mapsto r) = r$.

(11) $\prod(2 \mapsto r) = r \cdot r$.

(12) $\prod((n+1) \mapsto r) = \prod(n \mapsto r) \cdot r$.

(13) $j \neq 0$ and $r = 0$ iff $\prod(j \mapsto r) = 0$.

(14) If $r \neq 0$ and $j \leq i$, then $\prod((i -' j) \mapsto r) = \frac{\prod(i \mapsto r)}{\prod(j \mapsto r)}$.

(15) If $r \neq 0$ and $j \leq i$, then $r^{i -' j} = \frac{r^i}{r^j}$.

[1] The propositions (1)–(5) have been removed.

In the sequel a, b denote real numbers.
The following propositions are true:

(16) $^2\langle a,b\rangle = \langle a^{\mathbf{2}}, b^{\mathbf{2}}\rangle$.

(17) For every finite sequence F of elements of $\mathbb{R}$ such that $i \in \mathrm{dom}|F|$ and $a = F(i)$ holds $|F|(i) = |a|$.

(18) $|\langle a,b\rangle| = \langle |a|, |b|\rangle$.

(19) For all real numbers a, b, c, d such that $a \leq b$ and $c \leq d$ holds $|b-a|+|d-c| = (b-a)+(d-c)$.

(20) For all real numbers a, r such that $r > 0$ holds $a \in]a-r, a+r[$.

(21) For all real numbers a, r such that $r \geq 0$ holds $a \in [a-r, a+r]$.

(22) For all real numbers a, b such that $a < b$ holds $\inf]a,b[= a$ and $\sup]a,b[= b$.

(24)[2] For every bounded subset A of $\mathbb{R}$ holds $A \subseteq [\inf A, \sup A]$.

2. Topological Preliminaries

Let T be a topological structure and let A be a finite subset of T. Observe that $T \restriction A$ is finite.

One can check that there exists a topological space which is finite, non empty, and strict.

Let T be a topological structure. Observe that every subset of T which is empty is also connected.

Let T be a topological space. One can check that every subset of T which is finite is also compact.

One can prove the following two propositions:

(25) For all topological spaces S, T such that S and T are homeomorphic and S is connected holds T is connected.

(26) Let T be a topological space and F be a finite family of subsets of T. Suppose that for every subset X of T such that $X \in F$ holds X is compact. Then $\bigcup F$ is compact.

3. Points and Subsets in $\mathcal{E}^2_{\mathrm{T}}$

We now state a number of propositions:

(29)[3] For all sets A, B, C, D, a, b such that $A \subseteq B$ and $C \subseteq D$ holds $\prod[a \longmapsto A, b \longmapsto C] \subseteq \prod[a \longmapsto B, b \longmapsto D]$.

(30) For all subsets A, B of $\mathbb{R}$ holds $\prod[1 \longmapsto A, 2 \longmapsto B]$ is a subset of $\mathcal{E}^2_{\mathrm{T}}$.

(31) $|[0,a]| = |a|$ and $|[a,0]| = |a|$.

(32) For every point p of $\mathcal{E}^2$ and for every point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $p = 0_{\mathcal{E}^2_{\mathrm{T}}}$ and $p = q$ holds $q = \langle 0,0\rangle$ and $q_{\mathbf{1}} = 0$ and $q_{\mathbf{2}} = 0$.

(33) For all points p, q of $\mathcal{E}^2$ and for every point z of $\mathcal{E}^2_{\mathrm{T}}$ such that $p = 0_{\mathcal{E}^2_{\mathrm{T}}}$ and $q = z$ holds $\rho(p,q) = |z|$.

(34) $r \cdot p = [r \cdot p_{\mathbf{1}}, r \cdot p_{\mathbf{2}}]$.

(35) If $s = (1-r)\cdot p + r \cdot q$ and $s \neq p$ and $0 \leq r$, then $0 < r$.

(36) If $s = (1-r)\cdot p + r \cdot q$ and $s \neq q$ and $r \leq 1$, then $r < 1$.

[2] The proposition (23) has been removed.
[3] The propositions (27) and (28) have been removed.

(37) If $s \in \mathcal{L}(p,q)$ and $s \neq p$ and $s \neq q$ and $p_{\mathbf{1}} < q_{\mathbf{1}}$, then $p_{\mathbf{1}} < s_{\mathbf{1}}$ and $s_{\mathbf{1}} < q_{\mathbf{1}}$.

(38) If $s \in \mathcal{L}(p,q)$ and $s \neq p$ and $s \neq q$ and $p_{\mathbf{2}} < q_{\mathbf{2}}$, then $p_{\mathbf{2}} < s_{\mathbf{2}}$ and $s_{\mathbf{2}} < q_{\mathbf{2}}$.

(39) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ there exists a point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q_{\mathbf{1}} <$ W-bound(D) and $p \neq q$.

(40) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ there exists a point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q_{\mathbf{1}} >$ E-bound(D) and $p \neq q$.

(41) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ there exists a point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q_{\mathbf{2}} >$ N-bound(D) and $p \neq q$.

(42) For every point p of $\mathcal{E}^2_{\mathrm{T}}$ there exists a point q of $\mathcal{E}^2_{\mathrm{T}}$ such that $q_{\mathbf{2}} <$ S-bound(D) and $p \neq q$.

One can verify the following observations:

* every subset of $\mathcal{E}^2_{\mathrm{T}}$ which is convex and non empty is also connected,
* every subset of $\mathcal{E}^2_{\mathrm{T}}$ which is non horizontal is also non empty,
* every subset of $\mathcal{E}^2_{\mathrm{T}}$ which is non vertical is also non empty,
* every subset of $\mathcal{E}^2_{\mathrm{T}}$ which is region is also open and connected, and
* every subset of $\mathcal{E}^2_{\mathrm{T}}$ which is open and connected is also region.

One can check that every subset of $\mathcal{E}^2_{\mathrm{T}}$ which is empty is also horizontal and every subset of $\mathcal{E}^2_{\mathrm{T}}$ which is empty is also vertical.

Let us note that there exists a subset of $\mathcal{E}^2_{\mathrm{T}}$ which is non empty and convex.

Let a, b be points of $\mathcal{E}^2_{\mathrm{T}}$. Note that $\mathcal{L}(a,b)$ is convex and connected.

Let us observe that $\square_{\mathcal{E}^2}$ is connected.

Let us mention that every subset of $\mathcal{E}^2_{\mathrm{T}}$ which satisfies conditions of simple closed curve is also connected and compact.

The following propositions are true:

(43) $\mathcal{L}(\text{NE-corner}(P), \text{SE-corner}(P)) \subseteq \widetilde{\mathcal{L}}(\text{SpStSeq}\, P)$.

(44) $\mathcal{L}(\text{SW-corner}(P), \text{SE-corner}(P)) \subseteq \widetilde{\mathcal{L}}(\text{SpStSeq}\, P)$.

(45) $\mathcal{L}(\text{SW-corner}(P), \text{NW-corner}(P)) \subseteq \widetilde{\mathcal{L}}(\text{SpStSeq}\, P)$.

(46) For every subset C of $\mathcal{E}^2_{\mathrm{T}}$ holds $\{p; p$ ranges over points of $\mathcal{E}^2_{\mathrm{T}}$: $p_{\mathbf{1}} <$ W-bound$(C)\}$ is a non empty convex connected subset of $\mathcal{E}^2_{\mathrm{T}}$.

4. Balls as subsets of $\mathcal{E}^n_{\mathrm{T}}$

Next we state a number of propositions:

(47) If $e = q$ and $p \in \text{Ball}(e,r)$, then $q_{\mathbf{1}} - r < p_{\mathbf{1}}$ and $p_{\mathbf{1}} < q_{\mathbf{1}} + r$.

(48) If $e = q$ and $p \in \text{Ball}(e,r)$, then $q_{\mathbf{2}} - r < p_{\mathbf{2}}$ and $p_{\mathbf{2}} < q_{\mathbf{2}} + r$.

(49) If $p = e$, then $\prod[1 \longmapsto]p_{\mathbf{1}} - \frac{r}{\sqrt{2}}, p_{\mathbf{1}} + \frac{r}{\sqrt{2}}[, 2 \longmapsto]p_{\mathbf{2}} - \frac{r}{\sqrt{2}}, p_{\mathbf{2}} + \frac{r}{\sqrt{2}}[] \subseteq \text{Ball}(e,r)$.

(50) If $p = e$, then $\text{Ball}(e,r) \subseteq \prod[1 \longmapsto]p_{\mathbf{1}} - r, p_{\mathbf{1}} + r[, 2 \longmapsto]p_{\mathbf{2}} - r, p_{\mathbf{2}} + r[]$.

(51) If $P = \text{Ball}(e,r)$ and $p = e$, then $\text{proj1}^{\circ} P =]p_{\mathbf{1}} - r, p_{\mathbf{1}} + r[$.

(52) If $P = \text{Ball}(e,r)$ and $p = e$, then $\text{proj2}^{\circ} P =]p_{\mathbf{2}} - r, p_{\mathbf{2}} + r[$.

(53) If $D = \text{Ball}(e,r)$ and $p = e$, then W-bound$(D) = p_{\mathbf{1}} - r$.

(54) If $D = \text{Ball}(e,r)$ and $p = e$, then E-bound$(D) = p_{\mathbf{1}} + r$.

(55) If $D = \text{Ball}(e,r)$ and $p = e$, then S-bound$(D) = p_{\mathbf{2}} - r$.

(56) If $D = \mathrm{Ball}(e,r)$ and $p = e$, then N-bound$(D) = p_{\mathbf{2}} + r$.

(57) If $D = \mathrm{Ball}(e,r)$, then D is non horizontal.

(58) If $D = \mathrm{Ball}(e,r)$, then D is non vertical.

(59) For every point f of $\mathcal{E}^2$ and for every point x of $\mathcal{E}^2_{\mathrm{T}}$ such that $x \in \mathrm{Ball}(f,a)$ holds $[x_{\mathbf{1}} - 2 \cdot a, x_{\mathbf{2}}] \notin \mathrm{Ball}(f,a)$.

(60) Let X be a non empty compact subset of $\mathcal{E}^2_{\mathrm{T}}$ and p be a point of $\mathcal{E}^2$. If $p = 0_{\mathcal{E}^2_{\mathrm{T}}}$ and $a > 0$, then $X \subseteq \mathrm{Ball}(p, |\text{E-bound}(X)| + |\text{N-bound}(X)| + |\text{W-bound}(X)| + |\text{S-bound}(X)| + a)$.

(61) Let M be a Reflexive symmetric triangle non empty metric structure and z be a point of M. If $r < 0$, then $\mathrm{Sphere}(z,r) = \emptyset$.

(62) For every Reflexive discernible non empty metric structure M and for every point z of M holds $\mathrm{Sphere}(z,0) = \{z\}$.

(63) Let M be a Reflexive symmetric triangle non empty metric structure and z be a point of M. If $r < 0$, then $\overline{\mathrm{Ball}}(z,r) = \emptyset$.

(64) $\overline{\mathrm{Ball}}(z,0) = \{z\}$.

(65) For every subset A of M_{top} such that $A = \overline{\mathrm{Ball}}(z,r)$ holds A is closed.

(66) If $A = \overline{\mathrm{Ball}}(w,r)$, then A is closed.

(67) $\overline{\mathrm{Ball}}(z,r)$ is bounded.

(68) For every subset A of M_{top} such that $A = \mathrm{Sphere}(z,r)$ holds A is closed.

(69) If $A = \mathrm{Sphere}(w,r)$, then A is closed.

(70) $\mathrm{Sphere}(z,r)$ is bounded.

(71) If A is Bounded, then $\overline{A}$ is Bounded.

(72) For every non empty metric structure M holds M is bounded iff every subset of M is bounded.

(73) Let M be a Reflexive symmetric triangle non empty metric structure and X, Y be subsets of M. Suppose the carrier of $M = X \cup Y$ and M is non bounded and X is bounded. Then Y is non bounded.

(74) For all subsets X, Y of $\mathcal{E}^n_{\mathrm{T}}$ such that $n \geq 1$ and the carrier of $\mathcal{E}^n_{\mathrm{T}} = X \cup Y$ and X is Bounded holds Y is non Bounded.

(76)[4] If A is Bounded and B is Bounded, then $A \cup B$ is Bounded.

5. Topological Properties of Real Numbers Subsets

Let X be a non empty subset of $\mathbb{R}$. One can check that $\overline{X}$ is non empty.

Let D be a lower bounded subset of $\mathbb{R}$. One can check that $\overline{D}$ is lower bounded.

Let D be an upper bounded subset of $\mathbb{R}$. Observe that $\overline{D}$ is upper bounded.

The following propositions are true:

(77) For every non empty lower bounded subset D of $\mathbb{R}$ holds $\inf D = \inf \overline{D}$.

(78) For every non empty upper bounded subset D of $\mathbb{R}$ holds $\sup D = \sup \overline{D}$.

[4] The proposition (75) has been removed.

Let us mention that $\mathbb{R}^{\mathbf{1}}$ is T_2.

We now state three propositions:

(79) For every subset A of $\mathbb{R}$ and for every subset B of $\mathbb{R}^{\mathbf{1}}$ such that $A = B$ holds A is closed iff B is closed.

(80) For every subset A of $\mathbb{R}$ and for every subset B of $\mathbb{R}^{\mathbf{1}}$ such that $A = B$ holds $\overline{A} = \overline{B}$.

(81) For every subset A of $\mathbb{R}$ and for every subset B of $\mathbb{R}^{\mathbf{1}}$ such that $A = B$ holds A is compact iff B is compact.

Let us note that every subset of $\mathbb{R}$ which is finite is also compact.

Let a, b be real numbers. One can check that $[a,b]$ is compact.

The following proposition is true

(82) For all real numbers a, b holds $a \neq b$ iff $\overline{]a,b[} = [a,b]$.

Let us observe that there exists a subset of $\mathbb{R}$ which is non empty, finite, and bounded.

One can prove the following propositions:

(83) Let T be a topological structure, f be a real map of T, and g be a map from T into $\mathbb{R}^{\mathbf{1}}$. If $f = g$, then f is continuous iff g is continuous.

(84) Let A, B be subsets of $\mathbb{R}$ and f be a map from $[:\mathbb{R}^{\mathbf{1}}, \mathbb{R}^{\mathbf{1}}:]$ into $\mathcal{E}^2_{\mathrm{T}}$. If for all real numbers x, y holds $f(\langle x, y\rangle) = \langle x, y\rangle$, then $f^\circ[:A, B:] = \prod[1 \longmapsto A, 2 \longmapsto B]$.

(85) For every map f from $[:\mathbb{R}^{\mathbf{1}}, \mathbb{R}^{\mathbf{1}}:]$ into $\mathcal{E}^2_{\mathrm{T}}$ such that for all real numbers x, y holds $f(\langle x, y\rangle) = \langle x, y\rangle$ holds f is a homeomorphism.

(86) $[:\mathbb{R}^{\mathbf{1}}, \mathbb{R}^{\mathbf{1}}:]$ and $\mathcal{E}^2_{\mathrm{T}}$ are homeomorphic.

6. Bounded Subsets

The following propositions are true:

(87) For all compact subsets A, B of $\mathbb{R}$ holds $\prod[1 \longmapsto A, 2 \longmapsto B]$ is a compact subset of $\mathcal{E}^2_{\mathrm{T}}$.

(88) If P is Bounded and closed, then P is compact.

(89) If P is Bounded, then for every continuous real map g of $\mathcal{E}^2_{\mathrm{T}}$ holds $\overline{g^\circ P} \subseteq g^\circ \overline{P}$.

(90) $\mathrm{proj1}^\circ \overline{P} \subseteq \overline{\mathrm{proj1}^\circ P}$.

(91) $\mathrm{proj2}^\circ \overline{P} \subseteq \overline{\mathrm{proj2}^\circ P}$.

(92) If P is Bounded, then $\overline{\mathrm{proj1}^\circ P} = \mathrm{proj1}^\circ \overline{P}$.

(93) If P is Bounded, then $\overline{\mathrm{proj2}^\circ P} = \mathrm{proj2}^\circ \overline{P}$.

(94) If D is Bounded, then W-bound$(D) =$ W-bound$(\overline{D})$.

(95) If D is Bounded, then E-bound$(D) =$ E-bound$(\overline{D})$.

(96) If D is Bounded, then N-bound$(D) =$ N-bound$(\overline{D})$.

(97) If D is Bounded, then S-bound$(D) =$ S-bound$(\overline{D})$.

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