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Connectedness Conditions Using Polygonal Arcs

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Summary. A concept of special polygonal arc joining two different points is defined. Any two points in a ball can be connected by this kind of arc, and that is also true for any region in $\mathcal{E}^2_{\mathrm{T}}$.

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The articles [12], [14], [2], [8], [1], [15], [4], [3], [13], [11], [10], [9], [5], [6], and [7] provide the notation and terminology for this paper.

For simplicity, we use the following convention: P, P_1, P_2, R denote subsets of $\mathcal{E}^2_{\mathrm{T}}$, W denotes a non empty subset of $\mathcal{E}^2_{\mathrm{T}}$, p, p_1, p_2, q denote points of $\mathcal{E}^2_{\mathrm{T}}$, f, h denote finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, r denotes a real number, u denotes a point of $\mathcal{E}^2$, and n, i denote natural numbers.

The following proposition is true

(1) Let D be a non empty set, f be a finite sequence of elements of D, and p be an element of D. Then $(f \frown \langle p \rangle)_{\mathrm{len}\, f+1} = p$.

Let us consider P, p, q. We say that P is a special polygonal arc joining p and q if and only if:

(Def. 1) There exists f such that f is a special sequence and $P = \widetilde{\mathcal{L}}(f)$ and $p = f_1$ and $q = f_{\mathrm{len}\, f}$.

Let us consider P. We say that P is special polygon if and only if the condition (Def. 2) is satisfied.

(Def. 2) There exist p_1, p_2, P_1, P_2 such that
$p_1 \neq p_2$ and $p_1 \in P$ and $p_2 \in P$ and P_1 is a special polygonal arc joining p_1 and p_2 and P_2 is a special polygonal arc joining p_1 and p_2 and $P_1 \cap P_2 = \{p_1, p_2\}$ and $P = P_1 \cup P_2$.

We introduce P is a special polygon as a synonym of P is special polygon.

Let P be a subset of $\mathcal{E}^2_{\mathrm{T}}$. We say that P is region if and only if:

(Def. 3) P is open and connected.

We introduce P is a region as a synonym of P is region.

Next we state a number of propositions:

(2) If P is a special polygonal arc joining p and q, then P is a special polygonal arc.

(3) If W is a special polygonal arc joining p and q, then W is an arc from p to q.

(4) If W is a special polygonal arc joining p and q, then $p \in W$ and $q \in W$.

(5) If P is a special polygonal arc joining p and q, then $p \neq q$.

(6) If W is a special polygon, then W is a simple closed curve.

(7) Suppose $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $p \in \mathrm{Ball}(u,r)$ and $q \in \mathrm{Ball}(u,r)$ and $f = \langle p, [p_{\mathbf{1}}, \frac{p_{\mathbf{2}}+q_{\mathbf{2}}}{2}], q\rangle$. Then f is a special sequence and $f_1 = p$ and $f_{\mathrm{len}\, f} = q$ and $\widetilde{\mathcal{L}}(f)$ is a special polygonal arc joining p and q and $\widetilde{\mathcal{L}}(f) \subseteq \mathrm{Ball}(u,r)$.

(8) Suppose $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} = q_{\mathbf{2}}$ and $p \in \mathrm{Ball}(u,r)$ and $q \in \mathrm{Ball}(u,r)$ and $f = \langle p, [\frac{p_{\mathbf{1}}+q_{\mathbf{1}}}{2}, p_{\mathbf{2}}], q\rangle$. Then f is a special sequence and $f_1 = p$ and $f_{\mathrm{len}\, f} = q$ and $\widetilde{\mathcal{L}}(f)$ is a special polygonal arc joining p and q and $\widetilde{\mathcal{L}}(f) \subseteq \mathrm{Ball}(u,r)$.

(9) Suppose $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $p \in \mathrm{Ball}(u,r)$ and $q \in \mathrm{Ball}(u,r)$ and $[p_{\mathbf{1}}, q_{\mathbf{2}}] \in \mathrm{Ball}(u,r)$ and $f = \langle p, [p_{\mathbf{1}}, q_{\mathbf{2}}], q\rangle$. Then f is a special sequence and $f_1 = p$ and $f_{\mathrm{len}\, f} = q$ and $\widetilde{\mathcal{L}}(f)$ is a special polygonal arc joining p and q and $\widetilde{\mathcal{L}}(f) \subseteq \mathrm{Ball}(u,r)$.

(10) Suppose $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $p \in \mathrm{Ball}(u,r)$ and $q \in \mathrm{Ball}(u,r)$ and $[q_{\mathbf{1}}, p_{\mathbf{2}}] \in \mathrm{Ball}(u,r)$ and $f = \langle p, [q_{\mathbf{1}}, p_{\mathbf{2}}], q\rangle$. Then f is a special sequence and $f_1 = p$ and $f_{\mathrm{len}\, f} = q$ and $\widetilde{\mathcal{L}}(f)$ is a special polygonal arc joining p and q and $\widetilde{\mathcal{L}}(f) \subseteq \mathrm{Ball}(u,r)$.

(11) If $p \neq q$ and $p \in \mathrm{Ball}(u,r)$ and $q \in \mathrm{Ball}(u,r)$, then there exists P such that P is a special polygonal arc joining p and q and $P \subseteq \mathrm{Ball}(u,r)$.

(12) Suppose $p \neq f_1$ and $(f_1)_{\mathbf{2}} = p_{\mathbf{2}}$ and f is a special sequence and $p \in \mathcal{L}(f,1)$ and $h = \langle f_1, [\frac{(f_1)_{\mathbf{1}}+p_{\mathbf{1}}}{2}, (f_1)_{\mathbf{2}}], p\rangle$. Then

- (i) h is a special sequence,
- (ii) $h_1 = f_1$,
- (iii) $h_{\mathrm{len}\, h} = p$,
- (iv) $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining f_1 and p,
- (v) $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$, and
- (vi) $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction 1) \cup \mathcal{L}(f_1, p)$.

(13) Suppose $p \neq f_1$ and $(f_1)_{\mathbf{1}} = p_{\mathbf{1}}$ and f is a special sequence and $p \in \mathcal{L}(f,1)$ and $h = \langle f_1, [(f_1)_{\mathbf{1}}, \frac{(f_1)_{\mathbf{2}}+p_{\mathbf{2}}}{2}], p\rangle$. Then

- (i) h is a special sequence,
- (ii) $h_1 = f_1$,
- (iii) $h_{\mathrm{len}\, h} = p$,
- (iv) $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining f_1 and p,
- (v) $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$, and
- (vi) $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction 1) \cup \mathcal{L}(f_1, p)$.

(14) Suppose $p \neq f_1$ and f is a special sequence and $i \in \mathrm{dom}\, f$ and $i+1 \in \mathrm{dom}\, f$ and $i > 1$ and $p \in \mathcal{L}(f,i)$ and $p \neq f_i$ and $p \neq f_{i+1}$ and $h = (f \restriction i) ^\frown \langle p\rangle$. Then

- (i) h is a special sequence,
- (ii) $h_1 = f_1$,
- (iii) $h_{\mathrm{len}\, h} = p$,
- (iv) $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining f_1 and p,
- (v) $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$, and
- (vi) $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction i) \cup \mathcal{L}(f_i, p)$.

(15) Suppose $f_2 \neq f_1$ and f is a special sequence and $(f_2)_{\mathbf{2}} = (f_1)_{\mathbf{2}}$ and $h = \langle f_1, [\frac{(f_1)_{\mathbf{1}}+(f_2)_{\mathbf{1}}}{2}, (f_1)_{\mathbf{2}}], f_2\rangle$. Then h is a special sequence and $h_1 = f_1$ and $h_{\mathrm{len}\, h} = f_2$ and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining f_1 and f_2 and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$ and $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction 1) \cup \mathcal{L}(f_1, f_2)$ and $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f \restriction 2) \cup \mathcal{L}(f_2, f_2)$.

(16) Suppose $f_2 \neq f_1$ and f is a special sequence and $(f_2)_{\mathbf{1}} = (f_1)_{\mathbf{1}}$ and $h = \langle f_1, [(f_1)_{\mathbf{1}}, \frac{(f_1)_{\mathbf{2}}+(f_2)_{\mathbf{2}}}{2}], f_2\rangle$. Then h is a special sequence and $h_1 = f_1$ and $h_{\mathrm{len}\,h} = f_2$ and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining f_1 and f_2 and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$ and $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f{\restriction}1) \cup \mathcal{L}(f_1, f_2)$ and $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f{\restriction}2) \cup \mathcal{L}(f_2, f_2)$.

(17) Suppose $f_i \neq f_1$ and f is a special sequence and $i > 2$ and $i \in \operatorname{dom} f$ and $h = f{\restriction}i$. Then

(i) h is a special sequence,

(ii) $h_1 = f_1$,

(iii) $h_{\mathrm{len}\,h} = f_i$,

(iv) $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining f_1 and f_i,

(v) $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$, and

(vi) $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f{\restriction}i) \cup \mathcal{L}(f_i, f_i)$.

(18) Suppose $p \neq f_1$ and f is a special sequence and $p \in \mathcal{L}(f, n)$. Then there exists h such that

(i) h is a special sequence,

(ii) $h_1 = f_1$,

(iii) $h_{\mathrm{len}\,h} = p$,

(iv) $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining f_1 and p,

(v) $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$, and

(vi) $\widetilde{\mathcal{L}}(h) = \widetilde{\mathcal{L}}(f{\restriction}n) \cup \mathcal{L}(f_n, p)$.

(19) Suppose $p \neq f_1$ and f is a special sequence and $p \in \widetilde{\mathcal{L}}(f)$. Then there exists h such that h is a special sequence and $h_1 = f_1$ and $h_{\mathrm{len}\,h} = p$ and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining f_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f)$.

(20) Suppose that $p_{\mathbf{1}} = (f_{\mathrm{len}\,f})_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq (f_{\mathrm{len}\,f})_{\mathbf{2}}$ or $p_{\mathbf{1}} \neq (f_{\mathrm{len}\,f})_{\mathbf{1}}$ and $p_{\mathbf{2}} = (f_{\mathrm{len}\,f})_{\mathbf{2}}$ and $f_1 \notin \mathrm{Ball}(u,r)$ and $f_{\mathrm{len}\,f} \in \mathrm{Ball}(u,r)$ and $p \in \mathrm{Ball}(u,r)$ and f is a special sequence and $\mathcal{L}(f_{\mathrm{len}\,f}, p) \cap \widetilde{\mathcal{L}}(f) = \{f_{\mathrm{len}\,f}\}$ and $h = f \frown \langle p\rangle$. Then h is a special sequence and $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining f_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f) \cup \mathrm{Ball}(u,r)$.

(21) Suppose that $f_1 \notin \mathrm{Ball}(u,r)$ and $f_{\mathrm{len}\,f} \in \mathrm{Ball}(u,r)$ and $p \in \mathrm{Ball}(u,r)$ and $[p_{\mathbf{1}}, (f_{\mathrm{len}\,f})_{\mathbf{2}}] \in \mathrm{Ball}(u,r)$ and f is a special sequence and $p_{\mathbf{1}} \neq (f_{\mathrm{len}\,f})_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq (f_{\mathrm{len}\,f})_{\mathbf{2}}$ and $h = f \frown \langle [p_{\mathbf{1}}, (f_{\mathrm{len}\,f})_{\mathbf{2}}], p\rangle$ and $(\mathcal{L}(f_{\mathrm{len}\,f}, [p_{\mathbf{1}}, (f_{\mathrm{len}\,f})_{\mathbf{2}}]) \cup \mathcal{L}([p_{\mathbf{1}}, (f_{\mathrm{len}\,f})_{\mathbf{2}}], p)) \cap \widetilde{\mathcal{L}}(f) = \{f_{\mathrm{len}\,f}\}$. Then $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining f_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f) \cup \mathrm{Ball}(u,r)$.

(22) Suppose that $f_1 \notin \mathrm{Ball}(u,r)$ and $f_{\mathrm{len}\,f} \in \mathrm{Ball}(u,r)$ and $p \in \mathrm{Ball}(u,r)$ and $[(f_{\mathrm{len}\,f})_{\mathbf{1}}, p_{\mathbf{2}}] \in \mathrm{Ball}(u,r)$ and f is a special sequence and $p_{\mathbf{1}} \neq (f_{\mathrm{len}\,f})_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq (f_{\mathrm{len}\,f})_{\mathbf{2}}$ and $h = f \frown \langle [(f_{\mathrm{len}\,f})_{\mathbf{1}}, p_{\mathbf{2}}], p\rangle$ and $(\mathcal{L}(f_{\mathrm{len}\,f}, [(f_{\mathrm{len}\,f})_{\mathbf{1}}, p_{\mathbf{2}}]) \cup \mathcal{L}([(f_{\mathrm{len}\,f})_{\mathbf{1}}, p_{\mathbf{2}}], p)) \cap \widetilde{\mathcal{L}}(f) = \{f_{\mathrm{len}\,f}\}$. Then $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining f_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f) \cup \mathrm{Ball}(u,r)$.

(23) Suppose $f_1 \notin \mathrm{Ball}(u,r)$ and $f_{\mathrm{len}\,f} \in \mathrm{Ball}(u,r)$ and $p \in \mathrm{Ball}(u,r)$ and f is a special sequence and $p \notin \widetilde{\mathcal{L}}(f)$. Then there exists h such that $\widetilde{\mathcal{L}}(h)$ is a special polygonal arc joining f_1 and p and $\widetilde{\mathcal{L}}(h) \subseteq \widetilde{\mathcal{L}}(f) \cup \mathrm{Ball}(u,r)$.

(24) Let given R, p, p_1, p_2, P, r, u. Suppose $p \neq p_1$ and P is a special polygonal arc joining p_1 and p_2 and $P \subseteq R$ and $p \in \mathrm{Ball}(u,r)$ and $p_2 \in \mathrm{Ball}(u,r)$ and $\mathrm{Ball}(u,r) \subseteq R$. Then there exists a subset P_1 of $\mathcal{E}^2_{\mathrm{T}}$ such that P_1 is a special polygonal arc joining p_1 and p and $P_1 \subseteq R$.

In the sequel P, R are subsets of $\mathcal{E}^2_{\mathrm{T}}$.

We now state several propositions:

(25) Let given p. Suppose that

(i) R is a region, and

(ii) $P = \{q : q \neq p \wedge q \in R \wedge \neg \bigvee_{P_1 : \text{subset of } \mathcal{E}^2_\mathrm{T}} (P_1 \text{ is a special polygonal arc joining } p \text{ and } q \wedge P_1 \subseteq R)\}$.

Then P is open.

(26) Suppose that

(i) R is a region,

(ii) $p \in R$, and

(iii) $P = \{q : q = p \vee \bigvee_{P_1 : \text{subset of } \mathcal{E}^2_\mathrm{T}} (P_1 \text{ is a special polygonal arc joining } p \text{ and } q \wedge P_1 \subseteq R)\}$.

Then P is open.

(27) Suppose $p \in R$ and $P = \{q : q = p \vee \bigvee_{P_1 : \text{subset of } \mathcal{E}^2_\mathrm{T}} (P_1 \text{ is a special polygonal arc joining } p \text{ and } q \wedge P_1 \subseteq R)\}$. Then $P \subseteq R$.

(28) Suppose that

(i) R is a region,

(ii) $p \in R$, and

(iii) $P = \{q : q = p \vee \bigvee_{P_1 : \text{subset of } \mathcal{E}^2_\mathrm{T}} (P_1 \text{ is a special polygonal arc joining } p \text{ and } q \wedge P_1 \subseteq R)\}$.

Then $R \subseteq P$.

(29) Suppose that

(i) R is a region,

(ii) $p \in R$, and

(iii) $P = \{q : q = p \vee \bigvee_{P_1 : \text{subset of } \mathcal{E}^2_\mathrm{T}} (P_1 \text{ is a special polygonal arc joining } p \text{ and } q \wedge P_1 \subseteq R)\}$.

Then $R = P$.

(30) If R is a region and $p \in R$ and $q \in R$ and $p \neq q$, then there exists P such that P is a special polygonal arc joining p and q and $P \subseteq R$.

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