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Basic Properties of Connecting Points with Line Segments in $\mathcal{E}^2_{\mathrm{T}}$

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Summary. Some properties of line segments in 2-dimensional Euclidean space and some relations between line segments and balls are proved.

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The articles [14], [18], [2], [15], [11], [1], [19], [5], [3], [6], [4], [13], [16], [7], [17], [12], [8], [9], and [10] provide the notation and terminology for this paper.

1. Real Numbers Preliminaries

For simplicity, we adopt the following rules: p, p_1, p_2, p_3, q denote points of $\mathcal{E}^2_{\mathrm{T}}$, f, h denote finite sequences of elements of $\mathcal{E}^2_{\mathrm{T}}$, r, r_1, r_2, s, s_1, s_2 denote real numbers, u, u_1, u_2 denote points of $\mathcal{E}^2$, n, m, i, j denote natural numbers, and x, y, z denote sets.

We now state the proposition

(3)[1] If $r < s$, then $r < \frac{r+s}{2}$ and $\frac{r+s}{2} < s$.

2. Properties of Line Segments

The following propositions are true:

(6)[2] $1 \in \mathrm{dom}\langle x,y,z\rangle$ and $2 \in \mathrm{dom}\langle x,y,z\rangle$ and $3 \in \mathrm{dom}\langle x,y,z\rangle$.

(7) $(p_1+p_2)_{\mathbf{1}} = (p_1)_{\mathbf{1}} + (p_2)_{\mathbf{1}}$ and $(p_1+p_2)_{\mathbf{2}} = (p_1)_{\mathbf{2}} + (p_2)_{\mathbf{2}}$.

(8) $(p_1-p_2)_{\mathbf{1}} = (p_1)_{\mathbf{1}} - (p_2)_{\mathbf{1}}$ and $(p_1-p_2)_{\mathbf{2}} = (p_1)_{\mathbf{2}} - (p_2)_{\mathbf{2}}$.

(9) $(r \cdot p)_{\mathbf{1}} = r \cdot p_{\mathbf{1}}$ and $(r \cdot p)_{\mathbf{2}} = r \cdot p_{\mathbf{2}}$.

(10) If $p_1 = \langle r_1, s_1\rangle$ and $p_2 = \langle r_2, s_2\rangle$, then $p_1 + p_2 = \langle r_1+r_2, s_1+s_2\rangle$ and $p_1 - p_2 = \langle r_1 - r_2, s_1 - s_2\rangle$.

(11) If $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $p_{\mathbf{2}} = q_{\mathbf{2}}$, then $p = q$.

(12) If $u_1 = p_1$ and $u_2 = p_2$, then $\rho^2(u_1, u_2) = \sqrt{((p_1)_{\mathbf{1}} - (p_2)_{\mathbf{1}})^{\mathbf{2}} + ((p_1)_{\mathbf{2}} - (p_2)_{\mathbf{2}})^{\mathbf{2}}}$.

[1] The propositions (1) and (2) have been removed.
[2] The propositions (4) and (5) have been removed.

(13) The carrier of $\mathcal{E}^n_{\rm T}$ = the carrier of $\mathcal{E}^n$.

In the sequel r, r_1, r_2, s, s_1, s_2 are real numbers.
Next we state a number of propositions:

(15)[3] If $r_1 < s_1$, then $\{p_1 : (p_1)_{\mathbf{1}} = r \wedge r_1 \leq (p_1)_{\mathbf{2}} \wedge (p_1)_{\mathbf{2}} \leq s_1\} = \mathcal{L}([r,r_1],[r,s_1])$.

(16) If $r_1 < s_1$, then $\{p_1 : (p_1)_{\mathbf{2}} = r \wedge r_1 \leq (p_1)_{\mathbf{1}} \wedge (p_1)_{\mathbf{1}} \leq s_1\} = \mathcal{L}([r_1,r],[s_1,r])$.

(17) If $p \in \mathcal{L}([r,r_1],[r,s_1])$, then $p_{\mathbf{1}} = r$.

(18) If $p \in \mathcal{L}([r_1,r],[s_1,r])$, then $p_{\mathbf{2}} = r$.

(19) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} = q_{\mathbf{2}}$, then $[\frac{p_{\mathbf{1}}+q_{\mathbf{1}}}{2}, p_{\mathbf{2}}] \in \mathcal{L}(p,q)$.

(20) If $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$, then $[p_{\mathbf{1}}, \frac{p_{\mathbf{2}}+q_{\mathbf{2}}}{2}] \in \mathcal{L}(p,q)$.

(21) If $f = \langle p, p_1, q \rangle$ and $i \neq 0$ and $j > i+1$, then $\mathcal{L}(f,j) = \emptyset$.

(23)[4] If $f = \langle p_1, p_2, p_3 \rangle$, then $\widetilde{\mathcal{L}}(f) = \mathcal{L}(p_1,p_2) \cup \mathcal{L}(p_2,p_3)$.

(24) If $i \in \operatorname{dom} f$ and $j \in \operatorname{dom}(f \restriction i)$ and $j+1 \in \operatorname{dom}(f \restriction i)$, then $\mathcal{L}(f,j) = \mathcal{L}(f \restriction i, j)$.

(25) If $j \in \operatorname{dom} f$ and $j+1 \in \operatorname{dom} f$, then $\mathcal{L}(f \frown h, j) = \mathcal{L}(f,j)$.

(26) For every finite sequence f of elements of $\mathcal{E}^n_{\rm T}$ and for every natural number i holds $\mathcal{L}(f,i) \subseteq \widetilde{\mathcal{L}}(f)$.

(27) $\widetilde{\mathcal{L}}(f \restriction i) \subseteq \widetilde{\mathcal{L}}(f)$.

(28) Let r be a real number, p_1, p_2 be points of $\mathcal{E}^n_{\rm T}$, and u be a point of $\mathcal{E}^n$. If $p_1 \in \operatorname{Ball}(u,r)$ and $p_2 \in \operatorname{Ball}(u,r)$, then $\mathcal{L}(p_1,p_2) \subseteq \operatorname{Ball}(u,r)$.

(29) If $u = p_1$ and $p_1 = [r_1,s_1]$ and $p_2 = [r_2,s_2]$ and $p = [r_2,s_1]$ and $p_2 \in \operatorname{Ball}(u,r)$, then $p \in \operatorname{Ball}(u,r)$.

(30) If $[s,r_1] \in \operatorname{Ball}(u,r)$ and $[s,s_1] \in \operatorname{Ball}(u,r)$, then $[s, \frac{r_1+s_1}{2}] \in \operatorname{Ball}(u,r)$.

(31) If $[r_1,s] \in \operatorname{Ball}(u,r)$ and $[s_1,s] \in \operatorname{Ball}(u,r)$, then $[\frac{r_1+s_1}{2}, s] \in \operatorname{Ball}(u,r)$.

(32) If $r_1 \neq s_1$ and $s_2 \neq r_2$ and $[r_1,r_2] \in \operatorname{Ball}(u,r)$ and $[s_1,s_2] \in \operatorname{Ball}(u,r)$, then $[r_1,s_2] \in \operatorname{Ball}(u,r)$ or $[s_1,r_2] \in \operatorname{Ball}(u,r)$.

(33) Suppose $f_1 \notin \operatorname{Ball}(u,r)$ and $1 \leq m$ and $m \leq \operatorname{len} f - 1$ and $\mathcal{L}(f,m)$ meets $\operatorname{Ball}(u,r)$ and for every i such that $1 \leq i$ and $i \leq \operatorname{len} f - 1$ and $\mathcal{L}(f,i) \cap \operatorname{Ball}(u,r) \neq \emptyset$ holds $m \leq i$. Then $f_m \notin \operatorname{Ball}(u,r)$.

(34) For all q, p_2, p such that $q_{\mathbf{2}} = (p_2)_{\mathbf{2}}$ and $p_{\mathbf{2}} \neq (p_2)_{\mathbf{2}}$ holds $(\mathcal{L}(p_2,[(p_2)_{\mathbf{1}}, p_{\mathbf{2}}]) \cup \mathcal{L}([(p_2)_{\mathbf{1}}, p_{\mathbf{2}}],p)) \cap \mathcal{L}(q,p_2) = \{p_2\}$.

(35) For all q, p_2, p such that $q_{\mathbf{1}} = (p_2)_{\mathbf{1}}$ and $p_{\mathbf{1}} \neq (p_2)_{\mathbf{1}}$ holds $(\mathcal{L}(p_2,[p_{\mathbf{1}},(p_2)_{\mathbf{2}}]) \cup \mathcal{L}([p_{\mathbf{1}}, (p_2)_{\mathbf{2}}],p)) \cap \mathcal{L}(q,p_2) = \{p_2\}$.

(36) $\mathcal{L}(p,[p_{\mathbf{1}},q_{\mathbf{2}}]) \cap \mathcal{L}([p_{\mathbf{1}},q_{\mathbf{2}}],q) = \{[p_{\mathbf{1}},q_{\mathbf{2}}]\}$.

(37) $\mathcal{L}(p,[q_{\mathbf{1}},p_{\mathbf{2}}]) \cap \mathcal{L}([q_{\mathbf{1}},p_{\mathbf{2}}],q) = \{[q_{\mathbf{1}},p_{\mathbf{2}}]\}$.

(38) If $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$, then $\mathcal{L}(p,[p_{\mathbf{1}}, \frac{p_{\mathbf{2}}+q_{\mathbf{2}}}{2}]) \cap \mathcal{L}([p_{\mathbf{1}}, \frac{p_{\mathbf{2}}+q_{\mathbf{2}}}{2}],q) = \{[p_{\mathbf{1}}, \frac{p_{\mathbf{2}}+q_{\mathbf{2}}}{2}]\}$.

(39) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} = q_{\mathbf{2}}$, then $\mathcal{L}(p,[\frac{p_{\mathbf{1}}+q_{\mathbf{1}}}{2}, p_{\mathbf{2}}]) \cap \mathcal{L}([\frac{p_{\mathbf{1}}+q_{\mathbf{1}}}{2}, p_{\mathbf{2}}],q) = \{[\frac{p_{\mathbf{1}}+q_{\mathbf{1}}}{2}, p_{\mathbf{2}}]\}$.

(40) If $i > 2$ and $i \in \operatorname{dom} f$ and f is a special sequence, then $f \restriction i$ is a special sequence.

[3] The proposition (14) has been removed.
[4] The proposition (22) has been removed.

(41) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $f = \langle p, [p_{\mathbf{1}}, q_{\mathbf{2}}], q \rangle$, then $f_1 = p$ and $f_{\mathrm{len}\, f} = q$ and f is a special sequence.

(42) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $f = \langle p, [q_{\mathbf{1}}, p_{\mathbf{2}}], q \rangle$, then $f_1 = p$ and $f_{\mathrm{len}\, f} = q$ and f is a special sequence.

(43) If $p_{\mathbf{1}} = q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$ and $f = \langle p, [p_{\mathbf{1}}, \frac{p_{\mathbf{2}}+q_{\mathbf{2}}}{2}], q \rangle$, then $f_1 = p$ and $f_{\mathrm{len}\, f} = q$ and f is a special sequence.

(44) If $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} = q_{\mathbf{2}}$ and $f = \langle p, [\frac{p_{\mathbf{1}}+q_{\mathbf{1}}}{2}, p_{\mathbf{2}}], q \rangle$, then $f_1 = p$ and $f_{\mathrm{len}\, f} = q$ and f is a special sequence.

(45) If $i \in \mathrm{dom}\, f$ and $i+1 \in \mathrm{dom}\, f$, then $\widetilde{\mathcal{L}}(f \restriction (i+1)) = \widetilde{\mathcal{L}}(f \restriction i) \cup \mathcal{L}(f_i, f_{i+1})$.

(46) If $\mathrm{len}\, f \geq 2$ and $p \notin \widetilde{\mathcal{L}}(f)$, then for every n such that $1 \leq n$ and $n \leq \mathrm{len}\, f$ holds $f_n \neq p$.

(47) If $q \neq p$ and $\mathcal{L}(q, p) \cap \widetilde{\mathcal{L}}(f) = \{q\}$, then $p \notin \widetilde{\mathcal{L}}(f)$.

(48) Suppose f is a special sequence and $f_1 \notin \mathrm{Ball}(u, r)$ and $q \in \mathrm{Ball}(u, r)$ and $f_{\mathrm{len}\, f} \in \mathcal{L}(f, m)$ and $1 \leq m$ and $m+1 \leq \mathrm{len}\, f$ and $\mathcal{L}(f, m)$ meets $\mathrm{Ball}(u, r)$. Then $m+1 = \mathrm{len}\, f$.

(49) Suppose $p_1 \notin \mathrm{Ball}(u, r)$ and $q \in \mathrm{Ball}(u, r)$ and $p \in \mathrm{Ball}(u, r)$ and $p \notin \mathcal{L}(p_1, q)$ and $q_{\mathbf{1}} = p_{\mathbf{1}}$ and $q_{\mathbf{2}} \neq p_{\mathbf{2}}$ or $q_{\mathbf{1}} \neq p_{\mathbf{1}}$ and $q_{\mathbf{2}} = p_{\mathbf{2}}$ and $(p_1)_{\mathbf{1}} = q_{\mathbf{1}}$ or $(p_1)_{\mathbf{2}} = q_{\mathbf{2}}$. Then $\mathcal{L}(p_1, q) \cap \mathcal{L}(q, p) = \{q\}$.

(50) Suppose $p_1 \notin \mathrm{Ball}(u, r)$ and $p \in \mathrm{Ball}(u, r)$ and $[p_{\mathbf{1}}, q_{\mathbf{2}}] \in \mathrm{Ball}(u, r)$ and $q \in \mathrm{Ball}(u, r)$ and $[p_{\mathbf{1}}, q_{\mathbf{2}}] \notin \mathcal{L}(p_1, p)$ and $(p_1)_{\mathbf{1}} = p_{\mathbf{1}}$ and $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$. Then $(\mathcal{L}(p, [p_{\mathbf{1}}, q_{\mathbf{2}}]) \cup \mathcal{L}([p_{\mathbf{1}}, q_{\mathbf{2}}], q)) \cap \mathcal{L}(p_1, p) = \{p\}$.

(51) Suppose $p_1 \notin \mathrm{Ball}(u, r)$ and $p \in \mathrm{Ball}(u, r)$ and $[q_{\mathbf{1}}, p_{\mathbf{2}}] \in \mathrm{Ball}(u, r)$ and $q \in \mathrm{Ball}(u, r)$ and $[q_{\mathbf{1}}, p_{\mathbf{2}}] \notin \mathcal{L}(p_1, p)$ and $(p_1)_{\mathbf{2}} = p_{\mathbf{2}}$ and $p_{\mathbf{1}} \neq q_{\mathbf{1}}$ and $p_{\mathbf{2}} \neq q_{\mathbf{2}}$. Then $(\mathcal{L}(p, [q_{\mathbf{1}}, p_{\mathbf{2}}]) \cup \mathcal{L}([q_{\mathbf{1}}, p_{\mathbf{2}}], q)) \cap \mathcal{L}(p_1, p) = \{p\}$.

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