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Metric Spaces as Topological Spaces — Fundamental Concepts

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Summary. Some notions connected with metric spaces and the relationship between metric spaces and topological spaces. Compactness of topological spaces is transferred for the case of metric spaces [13]. Some basic theorems about translations of topological notions for metric spaces are proved. One-dimensional topological space $\mathbb{R}^1$ is introduced, too.

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The articles [18], [7], [21], [1], [20], [11], [22], [4], [6], [5], [9], [3], [12], [15], [8], [2], [14], [17], [16], [19], and [10] provide the notation and terminology for this paper.

For simplicity, we adopt the following rules: r is a real number, n is a natural number, a, b are real numbers, and T is a non empty topological space.

We now state the proposition

(1) Let T be a topological structure and F be a family of subsets of T. Then F is a cover of T if and only if the carrier of $T \subseteq \bigcup F$.

In the sequel A denotes a non empty subspace of T.

We now state three propositions:

(2) Every point of A is a point of T.

(3) If T is a T_2 space, then A is a T_2 space.

(4) For all subspaces A, B of T such that the carrier of $A \subseteq$ the carrier of B holds A is a subspace of B.

In the sequel P, Q denote subsets of T and p denotes a point of T.

The following propositions are true:

(5) $T \restriction P$ is a subspace of $T \restriction (P \cup Q)$ and $T \restriction Q$ is a subspace of $T \restriction (P \cup Q)$.

(6) Let P be a non empty subset of T. Suppose $p \in P$. Let Q be a neighbourhood of p, p' be a point of $T \restriction P$, and Q' be a subset of $T \restriction P$. If $Q' = Q \cap P$ and $p' = p$, then Q' is a neighbourhood of p'.

(7) Let A, B, C be topological spaces and f be a map from A into C. Suppose f is continuous and C is a subspace of B. Let h be a map from A into B. If $h = f$, then h is continuous.

(8) Let A be a topological space, B be a non empty topological space, f be a map from A into B, and C be a subspace of B. Suppose f is continuous. Let h be a map from A into C. If $h = f$, then h is continuous.

(9) Let A, B be topological spaces, f be a map from A into B, and C be a subset of B. Suppose f is continuous. Let h be a map from A into $B \restriction C$. If $h = f$, then h is continuous.

(10) Let X be a topological structure, Y be a non empty topological structure, K_0 be a subset of X, f be a map from X into Y, and g be a map from $X \restriction K_0$ into Y. If f is continuous and $g = f \restriction K_0$, then g is continuous.

In the sequel M denotes a non empty metric space and p denotes a point of M.

Let M be a metric space. A metric space is called a subspace of M if it satisfies the conditions (Def. 1).

(Def. 1)(i) The carrier of it $\subseteq$ the carrier of M, and

(ii) for all points x, y of it holds (the distance of it)(x, y) = (the distance of M)(x, y).

Let M be a metric space. Note that there exists a subspace of M which is strict.

Let M be a non empty metric space. Observe that there exists a subspace of M which is strict and non empty.

In the sequel A is a non empty subspace of M.

One can prove the following propositions:

(12)[1] Every point of A is a point of M.

(13) Let r be a real number, M be a metric space, A be a subspace of M, x be a point of M, and x' be a point of A. If $x = x'$, then $\mathrm{Ball}(x', r) = \mathrm{Ball}(x, r) \cap$ the carrier of A.

Let M be a non empty metric space and let A be a non empty subset of M. The functor $M \restriction A$ yielding a strict subspace of M is defined by:

(Def. 2) The carrier of $M \restriction A = A$.

Let M be a non empty metric space and let A be a non empty subset of M. Observe that $M \restriction A$ is non empty.

Let a, b be real numbers. Let us assume that $a \leq b$. The functor $[a, b]_{\mathrm{M}}$ yields a strict non empty subspace of the metric space of real numbers and is defined by the condition (Def. 3).

(Def. 3) Let P be a non empty subset of the metric space of real numbers. If $P = [a, b]$, then $[a, b]_{\mathrm{M}} =$ (the metric space of real numbers)$\restriction P$.

The following proposition is true

(14) If $a \leq b$, then the carrier of $[a, b]_{\mathrm{M}} = [a, b]$.

In the sequel F, G denote families of subsets of M.

Let M be a metric structure and let F be a family of subsets of M. We say that F is ball-family if and only if:

(Def. 4) For every set P such that $P \in F$ there exists a point p of M and there exists r such that $P = \mathrm{Ball}(p, r)$.

We introduce F is a family of balls as a synonym of F is ball-family. We say that F is a cover of M if and only if:

(Def. 5) The carrier of $M \subseteq \bigcup F$.

One can prove the following propositions:

[1] The proposition (11) has been removed.

(15) Let p, q be points of the metric space of real numbers and x, y be real numbers. If $x = p$ and $y = q$, then $\rho(p,q) = |x - y|$.

(16) Let M be a metric structure. Then the carrier of $M =$ the carrier of $M_{\rm top}$ and the topology of $M_{\rm top} =$ the open set family of M.

(19)[2] $A_{\rm top}$ is a subspace of $M_{\rm top}$.

(20) For every subset P of $\mathcal{E}^n_{\rm T}$ and for every non empty subset Q of $\mathcal{E}^n$ such that $P = Q$ holds $(\mathcal{E}^n_{\rm T}) \restriction P = (\mathcal{E}^n \restriction Q)_{\rm top}$.

(21) Let r be a real number, M be a triangle metric structure, p be a point of M, and P be a subset of $M_{\rm top}$. If $P = \mathrm{Ball}(p,r)$, then P is open.

(22) Let P be a subset of $M_{\rm top}$. Then P is open if and only if for every point p of M such that $p \in P$ there exists a real number r such that $r > 0$ and $\mathrm{Ball}(p,r) \subseteq P$.

Let M be a metric structure. We say that M is compact if and only if:

(Def. 6) $M_{\rm top}$ is compact.

One can prove the following proposition

(23) M is compact if and only if for every F such that F is a family of balls and a cover of M there exists G such that $G \subseteq F$ and G is a cover of M and finite.

The strict topological space $\mathbb{R}^{\mathbf{1}}$ is defined by:

(Def. 7) $\mathbb{R}^{\mathbf{1}} =$ (the metric space of real numbers)$_{\rm top}$.

Let us note that $\mathbb{R}^{\mathbf{1}}$ is non empty.
The following proposition is true

(24) The carrier of $\mathbb{R}^{\mathbf{1}} = \mathbb{R}$.

Let C be a set, let f be a partial function from C to the carrier of $\mathbb{R}^{\mathbf{1}}$, and let x be a set. One can verify that $f(x)$ is real.

Let a, b be real numbers. The functor $[a, b]_{\rm T}$ yields a strict non empty subspace of $\mathbb{R}^{\mathbf{1}}$ and is defined by:

(Def. 8) $[a, b]_{\rm T} = ([a, b]_{\rm M})_{\rm top}$.

The following propositions are true:

(25) If $a \leq b$, then the carrier of $[a, b]_{\rm T} = [a,b]$.

(26) If $a \leq b$, then for every subset P of $\mathbb{R}^{\mathbf{1}}$ such that $P = [a,b]$ holds $[a, b]_{\rm T} = \mathbb{R}^{\mathbf{1}} \restriction P$.

(27) $[0, 1]_{\rm T} = \mathbb{I}$.

$\mathbb{I}$ is a strict subspace of $\mathbb{R}^{\mathbf{1}}$.
Next we state the proposition

(28) Let f be a map from $\mathbb{R}^{\mathbf{1}}$ into $\mathbb{R}^{\mathbf{1}}$. Given real numbers a, b such that let x be a real number. Then $f(x) = a \cdot x + b$. Then f is continuous.

[2] The propositions (17) and (18) have been removed.

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