JOURNAL OF FORMALIZED MATHEMATICS
Volume 10, Released 1998, Published 2003
Inst. of Computer Science, Univ. of Białystok

# The Definition and Basic Properties of Topological Groups

Artur Korniłowicz
University of Białystok

MML Identifier: TOPGRP_1.

WWW: http://mizar.org/JFM/Vol10/topgrp_1.html

The articles [15], [7], [20], [21], [9], [4], [6], [3], [12], [11], [18], [19], [10], [13], [22], [8], [2], [1], [16], [23], [14], [5], and [17] provide the notation and terminology for this paper.

## 1. Preliminaries

For simplicity, we follow the rules: $S$ is a 1-sorted structure, $R$ is a non empty 1-sorted structure, $X$ is a subset of $R$, $T$ is a non empty topological structure, and $x$ is a set.

Let $X$ be a set. Observe that there exists a function from $X$ into $X$ which is one-to-one and onto.

We now state the proposition

(1) $\mathrm{rng}(\mathrm{id}_S) = \Omega_S$.

Let $R$ be a non empty 1-sorted structure. Note that $\mathrm{UNKNOWN}(\mathrm{id}_R)$ is one-to-one.

One can prove the following propositions:

(2) $\mathrm{UNKNOWN}(\mathrm{id}_R) = \mathrm{id}_R$.

(3) $(\mathrm{id}_R)^{-1}(X) = X$.

Let $S$ be a 1-sorted structure. One can check that there exists a map from $S$ into $S$ which is one-to-one and onto.

## 2. On the Groups

We adopt the following convention: $H$ denotes a non empty groupoid, $P$, $Q$, $P_1$, $Q_1$ denote subsets of $H$, and $h$ denotes an element of $H$.

We now state three propositions:

(4) If $P \subseteq P_1$ and $Q \subseteq Q_1$, then $P \cdot Q \subseteq P_1 \cdot Q_1$.

(5) If $P \subseteq Q$, then $P \cdot h \subseteq Q \cdot h$.

(6) If $P \subseteq Q$, then $h \cdot P \subseteq h \cdot Q$.

In the sequel $G$ denotes a group, $A$, $B$ denote subsets of $G$, and $a$ denotes an element of $G$.

The following propositions are true:

(7) $a \in A^{-1}$ iff $a^{-1} \in A$.

(8) $(A^{-1})^{-1} = A$.

(9) $A \subseteq B$ iff $A^{-1} \subseteq B^{-1}$.

(10) $\cdot_G^{-1\circ} A = A^{-1}$.

(11) $\cdot_G^{-1-1}(A) = A^{-1}$.

(12) $\cdot_G^{-1}$ is one-to-one.

(13) $\operatorname{rng} \cdot_G^{-1} =$ the carrier of $G$.

Let $G$ be a group. Observe that $\cdot_G^{-1}$ is one-to-one and onto.
One can prove the following propositions:

(14) $\cdot_G^{-1-1} = \cdot_G^{-1}$.

(15) (The multiplication of $H)^\circ[: P, Q :] = P \cdot Q$.

Let $G$ be a non empty groupoid and let $a$ be an element of $G$. The functor $a^*$ yields a map from $G$ into $G$ and is defined as follows:

(Def. 1) For every element $x$ of $G$ holds $a^*(x) = a \cdot x$.

The functor $\cdot a$ yields a map from $G$ into $G$ and is defined as follows:

(Def. 2) For every element $x$ of $G$ holds $(\cdot a)(x) = x \cdot a$.

Let $G$ be a group and let $a$ be an element of $G$. Observe that $a^*$ is one-to-one and onto and $\cdot a$ is one-to-one and onto.
The following four propositions are true:

(16) $(h^*)^\circ P = h \cdot P$.

(17) $(\cdot h)^\circ P = P \cdot h$.

(18) UNKNOWN$(a^*) = (a^{-1})^*$.

(19) UNKNOWN$(\cdot a) = \cdot a^{-1}$.

## 3. On the Topological Spaces

Let $T$ be a non empty topological structure. Observe that UNKNOWN$(\mathrm{id}_T)$ is continuous.
We now state the proposition

(20) $\mathrm{id}_T$ is a homeomorphism.

Let $T$ be a non empty topological space and let $p$ be a point of $T$. Observe that every neighbourhood of $p$ is non empty.
We now state the proposition

(21) For every non empty topological space $T$ and for every point $p$ of $T$ holds $\Omega_T$ is a neighbourhood of $p$.

Let $T$ be a non empty topological space and let $p$ be a point of $T$. Observe that there exists a neighbourhood of $p$ which is non empty and open.
One can prove the following propositions:

(22) Let $S, T$ be non empty topological spaces and $f$ be a map from $S$ into $T$. Suppose $f$ is open. Let $p$ be a point of $S$ and $P$ be a neighbourhood of $p$. Then there exists an open neighbourhood $R$ of $f(p)$ such that $R \subseteq f^\circ P$.

(23) Let $S$, $T$ be non empty topological spaces and $f$ be a map from $S$ into $T$. Suppose that for every point $p$ of $S$ and for every open neighbourhood $P$ of $p$ there exists a neighbourhood $R$ of $f(p)$ such that $R \subseteq f^{\circ}P$. Then $f$ is open.

(24) Let $S$, $T$ be non empty topological structures and $f$ be a map from $S$ into $T$. Then $f$ is a homeomorphism if and only if the following conditions are satisfied:

(i) $\operatorname{dom} f = \Omega_S$,

(ii) $\operatorname{rng} f = \Omega_T$,

(iii) $f$ is one-to-one, and

(iv) for every subset $P$ of $T$ holds $P$ is closed iff $f^{-1}(P)$ is closed.

(25) Let $S$, $T$ be non empty topological structures and $f$ be a map from $S$ into $T$. Then $f$ is a homeomorphism if and only if the following conditions are satisfied:

(i) $\operatorname{dom} f = \Omega_S$,

(ii) $\operatorname{rng} f = \Omega_T$,

(iii) $f$ is one-to-one, and

(iv) for every subset $P$ of $S$ holds $P$ is open iff $f^{\circ}P$ is open.

(26) Let $S$, $T$ be non empty topological structures and $f$ be a map from $S$ into $T$. Then $f$ is a homeomorphism if and only if the following conditions are satisfied:

(i) $\operatorname{dom} f = \Omega_S$,

(ii) $\operatorname{rng} f = \Omega_T$,

(iii) $f$ is one-to-one, and

(iv) for every subset $P$ of $T$ holds $P$ is open iff $f^{-1}(P)$ is open.

(27) Let $S$ be a topological space, $T$ be a non empty topological space, and $f$ be a map from $S$ into $T$. Then $f$ is continuous if and only if for every subset $P$ of $T$ holds $f^{-1}(\operatorname{Int} P) \subseteq \operatorname{Int}(f^{-1}(P))$.

Let $T$ be a non empty topological space. Observe that there exists a subset of $T$ which is non empty and dense.

The following two propositions are true:

(28) Let $S$, $T$ be non empty topological spaces, $f$ be a map from $S$ into $T$, and $A$ be a dense subset of $S$. If $f$ is a homeomorphism, then $f^{\circ}A$ is dense.

(29) Let $S$, $T$ be non empty topological spaces, $f$ be a map from $S$ into $T$, and $A$ be a dense subset of $T$. If $f$ is a homeomorphism, then $f^{-1}(A)$ is dense.

Let $S$, $T$ be non empty topological structures. Observe that every map from $S$ into $T$ which is homeomorphism is also onto, one-to-one, continuous, and open.

Let $T$ be a non empty topological structure. Note that there exists a map from $T$ into $T$ which is homeomorphism.

Let $T$ be a non empty topological structure and let $f$ be homeomorphism map from $T$ into $T$. One can check that UNKNOWN($f$) is homeomorphism.

## 4. The Group of Homeomorphisms

Let $T$ be a non empty topological structure. A map from $T$ into $T$ is said to be a homeomorphism of $T$ if:

(Def. 3) It is a homeomorphism.

Let $T$ be a non empty topological structure. Then $\mathrm{id}_T$ is a homeomorphism of $T$.

Let $T$ be a non empty topological structure. Note that every homeomorphism of $T$ is homeomorphism.

We now state two propositions:

(30) For every homeomorphism $f$ of $T$ holds UNKNOWN$(f)$ is a homeomorphism of $T$.

(31) For all homeomorphisms $f$, $g$ of $T$ holds $f \cdot g$ is a homeomorphism of $T$.

Let $T$ be a non empty topological structure. The group of homeomorphisms of $T$ is a strict groupoid and is defined by the conditions (Def. 4).

(Def. 4)(i) $x \in$ the carrier of the group of homeomorphisms of $T$ iff $x$ is a homeomorphism of $T$, and

(ii) for all homeomorphisms $f$, $g$ of $T$ holds (the multiplication of the group of homeomorphisms of $T$)$(f, g) = g \cdot f$.

Let $T$ be a non empty topological structure. Note that the group of homeomorphisms of $T$ is non empty.

We now state the proposition

(32) Let $f$, $g$ be homeomorphisms of $T$ and $a$, $b$ be elements of the group of homeomorphisms of $T$. If $f = a$ and $g = b$, then $a \cdot b = g \cdot f$.

Let $T$ be a non empty topological structure. One can check that the group of homeomorphisms of $T$ is group-like and associative.

Next we state two propositions:

(33) $\mathrm{id}_T = 1_{\text{the group of homeomorphisms of } T}$.

(34) Let $f$ be a homeomorphism of $T$ and $a$ be an element of the group of homeomorphisms of $T$. If $f = a$, then $a^{-1} =$ UNKNOWN$(f)$.

Let $T$ be a non empty topological structure. We say that $T$ is homogeneous if and only if:

(Def. 5) For all points $p$, $q$ of $T$ there exists a homeomorphism $f$ of $T$ such that $f(p) = q$.

Let us note that every non empty topological structure which is trivial is also homogeneous.

Let us observe that there exists a topological space which is strict, trivial, and non empty.

One can prove the following two propositions:

(35) Let $T$ be a homogeneous non empty topological space. If there exists a point $p$ of $T$ such that $\{p\}$ is closed, then $T$ is a $T_1$ space.

(36) Let $T$ be a homogeneous non empty topological space. Given a point $p$ of $T$ such that let $A$ be a subset of $T$. Suppose $A$ is open and $p \in A$. Then there exists a subset $B$ of $T$ such that $p \in B$ and $B$ is open and $\overline{B} \subseteq A$. Then $T$ is a $T_3$ space.

## 5. On the Topological Groups

We introduce topological group structures which are extensions of groupoid and topological structure and are systems

$\langle$ a carrier, a multiplication, a topology $\rangle$,

where the carrier is a set, the multiplication is a binary operation on the carrier, and the topology is a family of subsets of the carrier.

Let $A$ be a non empty set, let $R$ be a binary operation on $A$, and let $T$ be a family of subsets of $A$. Observe that $\langle A, R, T\rangle$ is non empty.

Let $x$ be a set, let $R$ be a binary operation on $\{x\}$, and let $T$ be a family of subsets of $\{x\}$. One can check that $\langle \{x\}, R, T\rangle$ is trivial.

One can check that every non empty groupoid which is trivial is also group-like, associative, and commutative.

Let $a$ be a set. One can check that $\{\{a\}\}_{\rm top}$ is trivial.

Let us note that there exists a topological group structure which is strict and non empty.

One can verify that there exists a non empty topological group structure which is strict, topological space-like, and trivial.

Let $G$ be a group-like associative non empty topological group structure. Then $\cdot_G^{-1}$ is a map from $G$ into $G$.

Let $G$ be a group-like associative non empty topological group structure. We say that $G$ is inverse-continuous if and only if:

(Def. 6) $\cdot_G^{-1}$ is continuous.

Let $G$ be a topological space-like topological group structure. We say that $G$ is continuous if and only if:

(Def. 7) For every map $f$ from $[:G, G:]$ into $G$ such that $f =$ the multiplication of $G$ holds $f$ is continuous.

Let us note that there exists a topological space-like group-like associative non empty topological group structure which is strict, commutative, trivial, inverse-continuous, and continuous.

A semi topological group is a topological space-like group-like associative non empty topological group structure.

A topological group is an inverse-continuous continuous semi topological group.

We now state several propositions:

(37) Let $T$ be a continuous non empty topological space-like topological group structure, $a$, $b$ be elements of $T$, and $W$ be a neighbourhood of $a \cdot b$. Then there exists an open neighbourhood $A$ of $a$ and there exists an open neighbourhood $B$ of $b$ such that $A \cdot B \subseteq W$.

(38) Let $T$ be a topological space-like non empty topological group structure. Suppose that for all elements $a$, $b$ of $T$ and for every neighbourhood $W$ of $a \cdot b$ there exists a neighbourhood $A$ of $a$ and there exists a neighbourhood $B$ of $b$ such that $A \cdot B \subseteq W$. Then $T$ is continuous.

(39) Let $T$ be an inverse-continuous semi topological group, $a$ be an element of $T$, and $W$ be a neighbourhood of $a^{-1}$. Then there exists an open neighbourhood $A$ of $a$ such that $A^{-1} \subseteq W$.

(40) Let $T$ be a semi topological group. Suppose that for every element $a$ of $T$ and for every neighbourhood $W$ of $a^{-1}$ there exists a neighbourhood $A$ of $a$ such that $A^{-1} \subseteq W$. Then $T$ is inverse-continuous.

(41) Let $T$ be a topological group, $a$, $b$ be elements of $T$, and $W$ be a neighbourhood of $a \cdot b^{-1}$. Then there exists an open neighbourhood $A$ of $a$ and there exists an open neighbourhood $B$ of $b$ such that $A \cdot B^{-1} \subseteq W$.

(42) Let $T$ be a semi topological group. Suppose that for all elements $a$, $b$ of $T$ and for every neighbourhood $W$ of $a \cdot b^{-1}$ there exists a neighbourhood $A$ of $a$ and there exists a neighbourhood $B$ of $b$ such that $A \cdot B^{-1} \subseteq W$. Then $T$ is a topological group.

Let $G$ be a continuous non empty topological space-like topological group structure and let $a$ be an element of $G$. One can verify that $a^*$ is continuous and $\cdot a$ is continuous.

The following two propositions are true:

(43) For every continuous semi topological group $G$ and for every element $a$ of $G$ holds $a^*$ is a homeomorphism of $G$.

(44) For every continuous semi topological group $G$ and for every element $a$ of $G$ holds $\cdot a$ is a homeomorphism of $G$.

Let $G$ be a continuous semi topological group and let $a$ be an element of $G$. Then $a^*$ is a homeomorphism of $G$. Then $\cdot a$ is a homeomorphism of $G$.

Next we state the proposition

(45) For every inverse-continuous semi topological group $G$ holds $\cdot_G^{-1}$ is a homeomorphism of $G$.

Let $G$ be an inverse-continuous semi topological group. Then $\cdot_G^{-1}$ is a homeomorphism of $G$.

One can check that every semi topological group which is continuous is also homogeneous.

Next we state two propositions:

(46) Let $G$ be a continuous semi topological group, $F$ be a closed subset of $G$, and $a$ be an element of $G$. Then $F \cdot a$ is closed.

(47) Let $G$ be a continuous semi topological group, $F$ be a closed subset of $G$, and $a$ be an element of $G$. Then $a \cdot F$ is closed.

Let $G$ be a continuous semi topological group, let $F$ be a closed subset of $G$, and let $a$ be an element of $G$. One can verify that $F \cdot a$ is closed and $a \cdot F$ is closed.

Next we state the proposition

(48) For every inverse-continuous semi topological group $G$ and for every closed subset $F$ of $G$ holds $F^{-1}$ is closed.

Let $G$ be an inverse-continuous semi topological group and let $F$ be a closed subset of $G$. One can check that $F^{-1}$ is closed.

One can prove the following propositions:

(49) Let $G$ be a continuous semi topological group, $O$ be an open subset of $G$, and $a$ be an element of $G$. Then $O \cdot a$ is open.

(50) Let $G$ be a continuous semi topological group, $O$ be an open subset of $G$, and $a$ be an element of $G$. Then $a \cdot O$ is open.

Let $G$ be a continuous semi topological group, let $A$ be an open subset of $G$, and let $a$ be an element of $G$. One can check that $A \cdot a$ is open and $a \cdot A$ is open.

The following proposition is true

(51) For every inverse-continuous semi topological group $G$ and for every open subset $O$ of $G$ holds $O^{-1}$ is open.

Let $G$ be an inverse-continuous semi topological group and let $A$ be an open subset of $G$. Observe that $A^{-1}$ is open.

One can prove the following two propositions:

(52) For every continuous semi topological group $G$ and for all subsets $A$, $O$ of $G$ such that $O$ is open holds $O \cdot A$ is open.

(53) For every continuous semi topological group $G$ and for all subsets $A$, $O$ of $G$ such that $O$ is open holds $A \cdot O$ is open.

Let $G$ be a continuous semi topological group, let $A$ be an open subset of $G$, and let $B$ be a subset of $G$. Note that $A \cdot B$ is open and $B \cdot A$ is open.

The following propositions are true:

(54) Let $G$ be an inverse-continuous semi topological group, $a$ be a point of $G$, and $A$ be a neighbourhood of $a$. Then $A^{-1}$ is a neighbourhood of $a^{-1}$.

(55) Let $G$ be a topological group, $a$ be a point of $G$, and $A$ be a neighbourhood of $a \cdot a^{-1}$. Then there exists an open neighbourhood $B$ of $a$ such that $B \cdot B^{-1} \subseteq A$.

(56) For every inverse-continuous semi topological group $G$ and for every dense subset $A$ of $G$ holds $A^{-1}$ is dense.

Let $G$ be an inverse-continuous semi topological group and let $A$ be a dense subset of $G$. Observe that $A^{-1}$ is dense.

Next we state two propositions:

(57) Let $G$ be a continuous semi topological group, $A$ be a dense subset of $G$, and $a$ be a point of $G$. Then $a \cdot A$ is dense.

(58) Let $G$ be a continuous semi topological group, $A$ be a dense subset of $G$, and $a$ be a point of $G$. Then $A \cdot a$ is dense.

Let $G$ be a continuous semi topological group, let $A$ be a dense subset of $G$, and let $a$ be a point of $G$. Observe that $A \cdot a$ is dense and $a \cdot A$ is dense.

We now state two propositions:

(59) Let $G$ be a topological group, $B$ be a basis of $1_G$, and $M$ be a dense subset of $G$. Then $\{V \cdot x; V$ ranges over subsets of $G$, $x$ ranges over points of $G$: $V \in B \wedge x \in M\}$ is a basis of $G$.

(60) Every topological group is a $T_3$ space.

Let us mention that every topological group is $T_3$.

## REFERENCES


[1] Józef Białas and Yatsuka Nakamura. Dyadic numbers and $T_4$ topological spaces. *Journal of Formalized Mathematics*, 7, 1995. http://mizar.org/JFM/Vol7/urysohn1.html.

[2] Leszek Borys. Paracompact and metrizable spaces. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/pcomps_1.html.

[3] Czesław Byliński. Binary operations. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/binop_1.html.

[4] Czesław Byliński. Functions and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_1.html.

[5] Czesław Byliński. Functions from a set to a set. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/funct_2.html.

[6] Czesław Byliński. Partial functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/partfun1.html.

[7] Czesław Byliński. Some basic properties of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/zfmisc_1.html.

[8] Agata Darmochwał. Compact spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/compts_1.html.

[9] Agata Darmochwał. Families of subsets, subspaces and mappings in topological spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/tops_2.html.

[10] Eugeniusz Kusak, Wojciech Leończuk, and Michał Muzalewski. Abelian groups, fields and vector spaces. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/vectsp_1.html.

[11] Michał Muzalewski. Categories of groups. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/grcat_1.html.

[12] Beata Padlewska. Families of sets. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/setfam_1.html.

[13] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Journal of Formalized Mathematics*, 1, 1989. http://mizar.org/JFM/Vol1/pre_topc.html.

[14] Alexander Yu. Shibakov and Andrzej Trybulec. The Cantor set. *Journal of Formalized Mathematics*, 7, 1995. http://mizar.org/JFM/Vol7/cantor_1.html.

[15] Andrzej Trybulec. Tarski Grothendieck set theory. *Journal of Formalized Mathematics*, Axiomatics, 1989. http://mizar.org/JFM/Axiomatics/tarski.html.

[16] Andrzej Trybulec. A Borsuk theorem on homotopy types. *Journal of Formalized Mathematics*, 3, 1991. http://mizar.org/JFM/Vol3/borsuk_1.html.

[17] Andrzej Trybulec. Baire spaces, Sober spaces. *Journal of Formalized Mathematics*, 9, 1997. http://mizar.org/JFM/Vol9/yellow_8.html.

[18] Wojciech A. Trybulec. Groups. *Journal of Formalized Mathematics*, 2, 1990. http://mizar.org/JFM/Vol2/group_1.html.

[19] Wojciech A. Trybulec. Subgroup and cosets of subgroups. *Journal of Formalized Mathematics*, 2, 1990. `http://mizar.org/JFM/Vol2/group_2.html`.

[20] Zinaida Trybulec. Properties of subsets. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/subset_1.html`.

[21] Edmund Woronowicz. Relations and their basic properties. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/relat_1.html`.

[22] Mirosław Wysocki and Agata Darmochwał. Subsets of topological spaces. *Journal of Formalized Mathematics*, 1, 1989. `http://mizar.org/JFM/Vol1/tops_1.html`.

[23] Mariusz Żynel and Adam Guzowski. $T_0$ topological spaces. *Journal of Formalized Mathematics*, 6, 1994. `http://mizar.org/JFM/Vol6/t_0topsp.html`.

*Received September 7, 1998*

*Published January 2, 2004*