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On Discrete and Almost Discrete Topological Spaces

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Summary. A topological space X is called *almost discrete* if every open subset of X is closed; equivalently, if every closed subset of X is open (comp. [6],[7]). Almost discrete spaces were investigated in Mizar formalism in [4]. We present here a few properties of such spaces supplementary to those given in [4].

Most interesting is the following characterization : *A topological space X is almost discrete iff every nonempty subset of X is not nowhere dense*. Hence, *X is non almost discrete iff there is an everywhere dense subset of X different from the carrier of X*. We have an analogous characterization of discrete spaces : *A topological space X is discrete iff every nonempty subset of X is not boundary*. Hence, *X is non discrete iff there is a dense subset of X different from the carrier of X*. It is well known that the class of all almost discrete spaces contains both the class of discrete spaces and the class of anti-discrete spaces (see e.g., [4]). Observations presented here show that the class of all almost discrete non discrete spaces is not contained in the class of anti-discrete spaces and the class of all almost discrete non anti-discrete spaces is not contained in the class of discrete spaces. Moreover, the class of almost discrete non discrete non anti-discrete spaces is nonempty. To analyse these interdependencies we use various examples of topological spaces constructed here in Mizar formalism.

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The articles [10], [11], [8], [1], [9], [12], [3], [4], [5], and [2] provide the notation and terminology for this paper.

1. Properties of Subsets of a Topological Space with Modified Topology

In this paper X is a non empty topological space and D is a subset of X.

We now state several propositions:

(1) Let B be a subset of X and C be a subset of the X modified w.r.t. D. If $B = C$, then if B is open, then C is open.

(2) Let B be a subset of X and C be a subset of the X modified w.r.t. D. If $B = C$, then if B is closed, then C is closed.

(3) For every subset C of the X modified w.r.t. D^{c} such that $C = D$ holds C is closed.

(4) For every subset C of the X modified w.r.t. D such that $C = D$ holds if D is dense, then C is dense and open.

(5) For every subset C of the X modified w.r.t. D such that $D \subseteq C$ holds if D is dense, then C is everywhere dense.

(6) Let C be a subset of the X modified w.r.t. D^{c}. If $C = D$, then if D is boundary, then C is boundary and closed.

(7) For every subset C of the X modified w.r.t. D^{c} such that $C \subseteq D$ holds if D is boundary, then C is nowhere dense.

2. Trivial Topological Spaces

Let Y be a non empty 1-sorted structure. Let us observe that Y is trivial if and only if:

(Def. 1) There exists an element d of Y such that the carrier of $Y = \{d\}$.

Let A be a non empty set. Note that $\langle A \rangle$ is non empty.

Let us observe that there exists a 1-sorted structure which is trivial, strict, and non empty and there exists a 1-sorted structure which is non trivial, strict, and non empty.

Let us note that there exists a topological structure which is trivial, strict, and non empty and there exists a topological structure which is non trivial, strict, and non empty.

We now state the proposition

(8) Let Y be a trivial non empty topological structure. Suppose the topology of Y is non empty. If Y is almost discrete, then Y is topological space-like.

Let us note that there exists a topological space which is trivial, strict, and non empty.

One can check that every non empty topological space which is trivial is also anti-discrete and discrete and every non empty topological space which is discrete and anti-discrete is also trivial.

One can check that there exists a topological space which is non trivial, strict, and non empty.

Let us note that every non empty topological space which is non discrete is also non trivial and every non empty topological space which is non anti-discrete is also non trivial.

3. Examples of Discrete and Anti-discrete Topological Spaces

Let D be a set. The functor 2^{D}_{*} yielding a family of subsets of D is defined by:

(Def. 2) $2^{D}_{*} = \{\emptyset, D\}$.

Let D be a set. Note that 2^{D}_{*} is non empty.

Let D be a set. The functor $\mathrm{ADTS}(D)$ yielding a topological structure is defined as follows:

(Def. 3) $\mathrm{ADTS}(D) = \langle D, 2^{D}_{*} \rangle$.

Let D be a set. One can check that $\mathrm{ADTS}(D)$ is strict, anti-discrete, and topological space-like.

Let D be a non empty set. One can check that $\mathrm{ADTS}(D)$ is non empty.

One can prove the following propositions:

(9) For every anti-discrete non empty topological space X holds the topological structure of $X = \mathrm{ADTS}(\text{the carrier of } X)$.

(10) Let X be a non empty topological space. Suppose the topological structure of $X =$ the topological structure of $\mathrm{ADTS}(\text{the carrier of } X)$. Then X is anti-discrete.

(11) Let X be an anti-discrete non empty topological space and A be a subset of X. Then

(i) if A is empty, then $\overline{A} = \emptyset$, and

(ii) if A is non empty, then $\overline{A} =$ the carrier of X.

(12) Let X be an anti-discrete non empty topological space and A be a subset of X. Then

(i) if $A \neq$ the carrier of X, then $\mathrm{Int} A = \emptyset$, and

(ii) if $A =$ the carrier of X, then $\mathrm{Int} A =$ the carrier of X.

(13) Let X be a non empty topological space. Suppose that for every subset A of X such that A is non empty holds $\overline{A}$ = the carrier of X. Then X is anti-discrete.

(14) Let X be a non empty topological space. Suppose that for every subset A of X such that $A \neq$ the carrier of X holds $\mathrm{Int}A = \emptyset$. Then X is anti-discrete.

(15) Let X be an anti-discrete non empty topological space and A be a subset of X. Then

(i) if $A \neq \emptyset$, then A is dense, and

(ii) if $A \neq$ the carrier of X, then A is boundary.

(16) Let X be a non empty topological space. Suppose that for every subset A of X such that $A \neq \emptyset$ holds A is dense. Then X is anti-discrete.

(17) Let X be a non empty topological space. Suppose that for every subset A of X such that $A \neq$ the carrier of X holds A is boundary. Then X is anti-discrete.

Let D be a set. Note that $\{D\}_{\mathrm{top}}$ is discrete.
One can prove the following propositions:

(18) For every discrete non empty topological space X holds the topological structure of X = $\{$the carrier of $X\}_{\mathrm{top}}$.

(19) Let X be a non empty topological space. Suppose the topological structure of X = the topological structure of $\{$the carrier of $X\}_{\mathrm{top}}$. Then X is discrete.

(20) For every discrete non empty topological space X and for every subset A of X holds $\overline{A} = A$ and $\mathrm{Int}A = A$.

(21) For every non empty topological space X such that for every subset A of X holds $\overline{A} = A$ holds X is discrete.

(22) For every non empty topological space X such that for every subset A of X holds $\mathrm{Int}A = A$ holds X is discrete.

(23) For every non empty set D holds $\mathrm{ADTS}(D) = \{D\}_{\mathrm{top}}$ iff there exists an element d_0 of D such that $D = \{d_0\}$.

Let us observe that there exists a topological space which is discrete, non anti-discrete, strict, and non empty and there exists a topological space which is anti-discrete, non discrete, strict, and non empty.

4. An Example of a Topological Space

Let D be a set, let F be a family of subsets of D, and let S be a set. Then $F \setminus S$ is a family of subsets of D.

Let D be a set and let d_0 be an element of D. The functor $\mathrm{STS}(D,d_0)$ yielding a topological structure is defined by:

(Def. 5)[1] $\mathrm{STS}(D,d_0) = \langle D, 2^D \setminus \{A; A \text{ ranges over subsets of } D\text{: } d_0 \in A \wedge A \neq D\}\rangle$.

Let D be a set and let d_0 be an element of D. Observe that $\mathrm{STS}(D,d_0)$ is strict and topological space-like.

Let D be a non empty set and let d_0 be an element of D. Observe that $\mathrm{STS}(D,d_0)$ is non empty.
In the sequel D is a non empty set and d_0 is an element of D.
We now state four propositions:

(24) For every subset A of $\mathrm{STS}(D,d_0)$ holds if $\{d_0\} \subseteq A$, then A is closed and if A is non empty and closed, then $\{d_0\} \subseteq A$.

[1] The definition (Def. 4) has been removed.

(25) Suppose $D \setminus \{d_0\}$ is non empty. Let A be a subset of $\mathrm{STS}(D, d_0)$. Then

(i) if $A = \{d_0\}$, then A is closed and boundary, and

(ii) if A is non empty, closed, and boundary, then $A = \{d_0\}$.

(26) For every subset A of $\mathrm{STS}(D, d_0)$ holds if $A \subseteq D \setminus \{d_0\}$, then A is open and if $A \neq D$ and A is open, then $A \subseteq D \setminus \{d_0\}$.

(27) Suppose $D \setminus \{d_0\}$ is non empty. Let A be a subset of $\mathrm{STS}(D, d_0)$. Then

(i) if $A = D \setminus \{d_0\}$, then A is open and dense, and

(ii) if $A \neq D$ and A is open and dense, then $A = D \setminus \{d_0\}$.

Let us mention that there exists a topological space which is non anti-discrete, non discrete, strict, and non empty.

Next we state several propositions:

(28) Let Y be a non empty topological space. Then the following statements are equivalent

(i) the topological structure of Y = the topological structure of $\mathrm{STS}(D, d_0)$,

(ii) the carrier of $Y = D$ and for every subset A of Y holds if $\{d_0\} \subseteq A$, then A is closed and if A is non empty and closed, then $\{d_0\} \subseteq A$.

(29) Let Y be a non empty topological space. Then the following statements are equivalent

(i) the topological structure of Y = the topological structure of $\mathrm{STS}(D, d_0)$,

(ii) the carrier of $Y = D$ and for every subset A of Y holds if $A \subseteq D \setminus \{d_0\}$, then A is open and if $A \neq D$ and A is open, then $A \subseteq D \setminus \{d_0\}$.

(30) Let Y be a non empty topological space. Then the following statements are equivalent

(i) the topological structure of Y = the topological structure of $\mathrm{STS}(D, d_0)$,

(ii) the carrier of $Y = D$ and for every non empty subset A of Y holds $\overline{A} = A \cup \{d_0\}$.

(31) Let Y be a non empty topological space. Then the following statements are equivalent

(i) the topological structure of Y = the topological structure of $\mathrm{STS}(D, d_0)$,

(ii) the carrier of $Y = D$ and for every subset A of Y such that $A \neq D$ holds $\mathrm{Int} A = A \setminus \{d_0\}$.

(32) $\mathrm{STS}(D, d_0) = \mathrm{ADTS}(D)$ iff $D = \{d_0\}$.

(33) $\mathrm{STS}(D, d_0) = \{D\}_{\mathrm{top}}$ iff $D = \{d_0\}$.

(34) Let D be a non empty set, d_0 be an element of D, and A be a subset of $\mathrm{STS}(D, d_0)$. If $A = \{d_0\}$, then $\{D\}_{\mathrm{top}}$ = the $\mathrm{STS}(D, d_0)$ modified w.r.t. A.

5. Discrete and Almost Discrete Spaces

Let X be a non empty topological space. Let us observe that X is discrete if and only if:

(Def. 6) For every non empty subset A of X holds A is not boundary.

The following proposition is true

(35) X is discrete iff for every subset A of X such that $A \neq$ the carrier of X holds A is not dense.

Let us note that every non empty topological space which is non almost discrete is also non discrete and non anti-discrete.

Let X be a non empty topological space. Let us observe that X is almost discrete if and only if:

(Def. 7) For every non empty subset A of X holds A is not nowhere dense.

Next we state three propositions:

(36) X is almost discrete if and only if for every subset A of X such that $A \neq$ the carrier of X holds A is not everywhere dense.

(37) X is non almost discrete iff there exists a non empty subset of X which is boundary and closed.

(38) X is non almost discrete if and only if there exists a subset A of X such that $A \neq$ the carrier of X and A is dense and open.

Let us observe that there exists a topological space which is almost discrete, non discrete, non anti-discrete, strict, and non empty.

The following proposition is true

(39) For every non empty set C and for every element c_0 of C holds $C \setminus \{c_0\}$ is non empty iff $\mathrm{STS}(C, c_0)$ is non almost discrete.

Let us mention that there exists a topological space which is non almost discrete, strict, and non empty.

Next we state two propositions:

(40) For every non empty subset A of X such that A is boundary holds the X modified w.r.t. A^{c} is non almost discrete.

(41) Let A be a subset of X. Suppose $A \neq$ the carrier of X and A is dense. Then the X modified w.r.t. A is non almost discrete.

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