JOURNAL OF FORMALIZED MATHEMATICS
Volume 4, Released 1992, Published 2003
Inst. of Computer Science, Univ. of Białystok

Completeness of the Lattices of Domains of a Topological Space[1]

Zbigniew Karno
Warsaw University
Białystok

Toshihiko Watanabe
Shinshu University
Nagano

Summary. Let T be a topological space and let A be a subset of T. Recall that A is said to be a *domain* in T provided $\mathrm{Int}\overline{A} \subseteq A \subseteq \overline{\mathrm{Int}A}$ (see [18] and comp. [9]). This notion is a simple generalization of the notions of open and closed domains in T (see [18]). Our main result is concerned with an extension of the following well–known theorem (see e.g. [2], [12], [8]). For a given topological space the Boolean lattices of all its closed domains and all its open domains are complete. It is proved here, using Mizar System, that *the complemented lattice of all domains of a given topological space is complete*, too (comp. [17]).

It is known that both the lattice of open domains and the lattice of closed domains are sublattices of the lattice of all domains [17]. However, the following two problems remain open.

Problem 1. Let L be a sublattice of the lattice of all domains. Suppose L is complete, is smallest with respect to inclusion, and contains as sublattices the lattice of all closed domains and the lattice of all open domains. Must L be equal to the lattice of all domains ?

A domain in T is said to be a *Borel domain* provided it is a Borel set. Of course every open (closed) domain is a Borel domain. It can be proved that all Borel domains form a sublattice of the lattice of domains.

Problem 2. Let L be a sublattice of the lattice of all domains. Suppose L is smallest with respect to inclusion and contains as sublattices the lattice of all closed domains and the lattice of all open domains. Must L be equal to the lattice of all Borel domains ?

Note that in the beginning the closure and the interior operations for families of subsets of topological spaces are introduced and their important properties are presented (comp. [11], [10], [12]). Using these notions, certain properties of domains, closed domains and open domains are studied (comp. [10], [8]).

MML Identifier: `TDLAT_2`.

WWW: `http://mizar.org/JFM/Vol4/tdlat_2.html`

The articles [15], [16], [13], [5], [7], [14], [18], [6], [3], [4], [19], [1], and [17] provide the notation and terminology for this paper.

1. Preliminary Theorems about Subsets of Topological Spaces

In this paper T is a non empty topological space.

We now state several propositions:

[1]This paper was done while the second author was visiting the Institute of Mathematics of Warsaw University in Białystok.

(1) For every subset A of T holds $\mathrm{Int}\overline{\mathrm{Int}A} \subseteq \mathrm{Int}\overline{A}$ and $\mathrm{Int}\overline{\mathrm{Int}A} \subseteq \overline{\mathrm{Int}A}$.

(2) For every subset A of T holds $\overline{\mathrm{Int}A} \subseteq \overline{\mathrm{Int}\overline{A}}$ and $\mathrm{Int}\overline{A} \subseteq \overline{\mathrm{Int}\overline{A}}$.

(3) For every subset A of T and for every subset B of T such that B is closed holds if $\overline{\mathrm{Int}(A \cap B)} = A$, then $A \subseteq B$.

(4) For every subset A of T and for every subset B of T such that A is open holds if $\mathrm{Int}\overline{A \cup B} = B$, then $A \subseteq B$.

(5) For every subset A of T such that $A \subseteq \overline{\mathrm{Int}A}$ holds $A \cup \mathrm{Int}\overline{A} \subseteq \overline{\mathrm{Int}(A \cup \mathrm{Int}\overline{A})}$.

(6) For every subset A of T such that $\mathrm{Int}\overline{A} \subseteq A$ holds $\mathrm{Int}\overline{A \cap \overline{\mathrm{Int}A}} \subseteq A \cap \overline{\mathrm{Int}A}$.

2. The Closure and the Interior Operations for Families of Subsets of a Topological Space

In the sequel T denotes a non empty topological space.

Let us consider T and let F be a family of subsets of T. We introduce $\overline{F}$ as a synonym of $\mathrm{clf}\,F$.

We now state several propositions:

(7) For every family F of subsets of T holds $\overline{F} = \{A; A$ ranges over subsets of T: $\bigvee_{B\,:\,\text{subset of }T}\ (A = \overline{B}\ \wedge\ B \in F)\}$.

(8) For every family F of subsets of T holds $\overline{F} = \overline{\overline{F}}$.

(9) For every family F of subsets of T holds $F = \emptyset$ iff $\overline{F} = \emptyset$.

(10) For all families F, G of subsets of T holds $\overline{F \cap G} \subseteq \overline{F} \cap \overline{G}$.

(11) For all families F, G of subsets of T holds $\overline{F} \setminus \overline{G} \subseteq \overline{F \setminus G}$.

(12) For every family F of subsets of T and for every subset A of T such that $A \in F$ holds $\bigcap\overline{F} \subseteq \overline{A}$ and $\overline{A} \subseteq \bigcup\overline{F}$.

(13) For every family F of subsets of T holds $\bigcap F \subseteq \bigcap\overline{F}$.

(14) For every family F of subsets of T holds $\overline{\bigcap F} \subseteq \bigcap\overline{F}$.

(15) For every family F of subsets of T holds $\bigcup\overline{F} \subseteq \overline{\bigcup F}$.

Let us consider T and let F be a family of subsets of T. The functor $\mathrm{Int}F$ yields a family of subsets of T and is defined as follows:

(Def. 1) For every subset A of T holds $A \in \mathrm{Int}F$ iff there exists a subset B of T such that $A = \mathrm{Int}B$ and $B \in F$.

We now state a number of propositions:

(16) For every family F of subsets of T holds $\mathrm{Int}F = \{A; A$ ranges over subsets of T: $\bigvee_{B\,:\,\text{subset of }T}\ (A = \mathrm{Int}B\ \wedge\ B \in F)\}$.

(17) For every family F of subsets of T holds $\mathrm{Int}F = \mathrm{Int}\,\mathrm{Int}F$.

(18) For every family F of subsets of T holds $\mathrm{Int}F$ is open.

(19) For every family F of subsets of T holds $F = \emptyset$ iff $\mathrm{Int}F = \emptyset$.

(20) For every subset A of T and for every family F of subsets of T such that $F = \{A\}$ holds $\mathrm{Int}F = \{\mathrm{Int}A\}$.

(21) For all families F, G of subsets of T such that $F \subseteq G$ holds $\mathrm{Int}F \subseteq \mathrm{Int}G$.

(22) For all families F, G of subsets of T holds $\mathrm{Int}(F \cup G) = \mathrm{Int}\,F \cup \mathrm{Int}\,G$.

(23) For all families F, G of subsets of T holds $\mathrm{Int}(F \cap G) \subseteq \mathrm{Int}\,F \cap \mathrm{Int}\,G$.

(24) For all families F, G of subsets of T holds $\mathrm{Int}\,F \setminus \mathrm{Int}\,G \subseteq \mathrm{Int}(F \setminus G)$.

(25) For every family F of subsets of T and for every subset A of T such that $A \in F$ holds $\mathrm{Int}\,A \subseteq \bigcup \mathrm{Int}\,F$ and $\bigcap \mathrm{Int}\,F \subseteq \mathrm{Int}\,A$.

(26) For every family F of subsets of T holds $\bigcup \mathrm{Int}\,F \subseteq \bigcup F$.

(27) For every family F of subsets of T holds $\bigcap \mathrm{Int}\,F \subseteq \bigcap F$.

(28) For every family F of subsets of T holds $\bigcup \mathrm{Int}\,F \subseteq \mathrm{Int} \bigcup F$.

(29) For every family F of subsets of T holds $\mathrm{Int} \bigcap F \subseteq \bigcap \mathrm{Int}\,F$.

(30) For every family F of subsets of T such that F is finite holds $\mathrm{Int} \bigcap F = \bigcap \mathrm{Int}\,F$.

In the sequel F is a family of subsets of T.

One can prove the following propositions:

(31) $\overline{\mathrm{Int}\,F} = \{A; A$ ranges over subsets of T: $\bigvee_{B:\text{subset of } T} (A = \overline{\mathrm{Int}\,B} \wedge B \in F)\}$.

(32) $\mathrm{Int}\,\overline{F} = \{A; A$ ranges over subsets of T: $\bigvee_{B:\text{subset of } T} (A = \mathrm{Int}\,\overline{B} \wedge B \in F)\}$.

(33) $\overline{\mathrm{Int}\,\overline{F}} = \{A; A$ ranges over subsets of T: $\bigvee_{B:\text{subset of } T} (A = \overline{\mathrm{Int}\,\overline{B}} \wedge B \in F)\}$.

(34) $\mathrm{Int}\,\overline{\mathrm{Int}\,F} = \{A; A$ ranges over subsets of T: $\bigvee_{B:\text{subset of } T} (A = \mathrm{Int}\,\overline{\mathrm{Int}\,B} \wedge B \in F)\}$.

(35) $\overline{\mathrm{Int}\,\overline{\mathrm{Int}\,F}} = \overline{\mathrm{Int}\,F}$.

(36) $\mathrm{Int}\,\overline{\mathrm{Int}\,\overline{F}} = \mathrm{Int}\,\overline{F}$.

(37) $\bigcup \mathrm{Int}\,\overline{F} \subseteq \bigcup \overline{\mathrm{Int}\,\overline{F}}$.

(38) $\bigcap \mathrm{Int}\,\overline{F} \subseteq \bigcap \overline{\mathrm{Int}\,\overline{F}}$.

(39) $\bigcup \overline{\mathrm{Int}\,F} \subseteq \bigcup \overline{\mathrm{Int}\,\overline{F}}$.

(40) $\bigcap \overline{\mathrm{Int}\,F} \subseteq \bigcap \overline{\mathrm{Int}\,\overline{F}}$.

(41) $\bigcup \mathrm{Int}\,\overline{\mathrm{Int}\,F} \subseteq \bigcup \mathrm{Int}\,\overline{F}$.

(42) $\bigcap \mathrm{Int}\,\overline{\mathrm{Int}\,F} \subseteq \bigcap \mathrm{Int}\,\overline{F}$.

(43) $\bigcup \mathrm{Int}\,\overline{\mathrm{Int}\,F} \subseteq \bigcup \overline{\mathrm{Int}\,F}$.

(44) $\bigcap \mathrm{Int}\,\overline{\mathrm{Int}\,F} \subseteq \bigcap \overline{\mathrm{Int}\,F}$.

(45) $\bigcup \overline{\mathrm{Int}\,\overline{F}} \subseteq \bigcup \overline{F}$.

(46) $\bigcap \overline{\mathrm{Int}\,\overline{F}} \subseteq \bigcap \overline{F}$.

(47) $\bigcup \mathrm{Int}\,F \subseteq \bigcup \mathrm{Int}\,\overline{\mathrm{Int}\,F}$.

(48) $\bigcap \mathrm{Int}\,F \subseteq \bigcap \mathrm{Int}\,\overline{\mathrm{Int}\,F}$.

(49) $\bigcup \overline{\mathrm{Int}\,F} \subseteq \overline{\mathrm{Int} \bigcup F}$.

(50) $\overline{\mathrm{Int} \bigcap F} \subseteq \bigcap \overline{\mathrm{Int}\,F}$.

(51) $\bigcup \mathrm{Int}\,\overline{F} \subseteq \mathrm{Int}\,\overline{\bigcup F}$.

(52) $\mathrm{Int}\,\overline{\bigcap F} \subseteq \bigcap \mathrm{Int}\,\overline{F}$.

(53) $\bigcup \overline{\mathrm{Int}\overline{F}} \subseteq \overline{\mathrm{Int}\overline{\bigcup F}}$.

(54) $\overline{\mathrm{Int}\overline{\bigcap F}} \subseteq \bigcap \overline{\mathrm{Int}\overline{F}}$.

(55) $\bigcup \mathrm{Int}\overline{\mathrm{Int}F} \subseteq \mathrm{Int}\overline{\mathrm{Int}\bigcup F}$.

(56) $\mathrm{Int}\overline{\mathrm{Int}\bigcap F} \subseteq \bigcap \mathrm{Int}\overline{\mathrm{Int}F}$.

(57) Let F be a family of subsets of T. Suppose that for every subset A of T such that $A \in F$ holds $A \subseteq \overline{\mathrm{Int}A}$. Then $\bigcup F \subseteq \overline{\mathrm{Int}\bigcup F}$ and $\overline{\bigcup F} = \overline{\mathrm{Int}\overline{\bigcup F}}$.

(58) Let F be a family of subsets of T. Suppose that for every subset A of T such that $A \in F$ holds $\mathrm{Int}\overline{A} \subseteq A$. Then $\mathrm{Int}\overline{\bigcap F} \subseteq \bigcap F$ and $\mathrm{Int}\overline{\mathrm{Int}\bigcap F} = \mathrm{Int}\bigcap F$.

3. Selected Properties of Domains of a Topological Space

In the sequel T denotes a non empty topological space.

Next we state several propositions:

(59) For all subsets A, B of T such that B is condensed holds $\mathrm{Int}\overline{A\cup B} \cup (A \cup B) = B$ iff $A \subseteq B$.

(60) For all subsets A, B of T such that A is condensed holds $\overline{\mathrm{Int}(A\cap B)} \cap (A \cap B) = A$ iff $A \subseteq B$.

(61) Let A, B be subsets of T. Suppose A is closed condensed and B is closed condensed. Then $\mathrm{Int}A \subseteq \mathrm{Int}B$ if and only if $A \subseteq B$.

(62) For all subsets A, B of T such that A is open condensed and B is open condensed holds $\overline{A} \subseteq \overline{B}$ iff $A \subseteq B$.

(63) For all subsets A, B of T such that A is closed condensed holds if $A \subseteq B$, then $\overline{\mathrm{Int}(A\cap B)} = A$.

(64) For all subsets A, B of T such that B is open condensed holds if $A \subseteq B$, then $\mathrm{Int}\overline{A\cup B} = B$.

Let us consider T and let I_1 be a family of subsets of T. We say that I_1 is domains-family if and only if:

(Def. 2) For every subset A of T such that $A \in I_1$ holds A is condensed.

Next we state several propositions:

(65) For every family F of subsets of T holds $F \subseteq$ the domains of T iff F is domains-family.

(66) For every family F of subsets of T such that F is domains-family holds $\bigcup F \subseteq \overline{\mathrm{Int}\bigcup F}$ and $\overline{\bigcup F} = \overline{\mathrm{Int}\overline{\bigcup F}}$.

(67) For every family F of subsets of T such that F is domains-family holds $\mathrm{Int}\overline{\bigcap F} \subseteq \bigcap F$ and $\mathrm{Int}\overline{\mathrm{Int}\bigcap F} = \mathrm{Int}\bigcap F$.

(68) For every family F of subsets of T such that F is domains-family holds $\bigcup F \cup \mathrm{Int}\overline{\bigcup F}$ is condensed.

(69) Let F be a family of subsets of T. Then

(i) for every subset B of T such that $B \in F$ holds $B \subseteq \bigcup F \cup \mathrm{Int}\overline{\bigcup F}$, and

(ii) for every subset A of T such that A is condensed holds if for every subset B of T such that $B \in F$ holds $B \subseteq A$, then $\bigcup F \cup \mathrm{Int}\overline{\bigcup F} \subseteq A$.

(70) For every family F of subsets of T such that F is domains-family holds $\bigcap F \cap \overline{\mathrm{Int}\bigcap F}$ is condensed.

(71) Let F be a family of subsets of T. Then

(i) for every subset B of T such that $B \in F$ holds $\bigcap F \cap \overline{\mathrm{Int} \bigcap F} \subseteq B$, and

(ii) $F = \emptyset$ or for every subset A of T such that A is condensed holds if for every subset B of T such that $B \in F$ holds $A \subseteq B$, then $A \subseteq \bigcap F \cap \overline{\mathrm{Int} \bigcap F}$.

Let us consider T and let I_1 be a family of subsets of T. We say that I_1 is closed-domains-family if and only if:

(Def. 3) For every subset A of T such that $A \in I_1$ holds A is closed condensed.

We now state several propositions:

(72) Let F be a family of subsets of T. Then $F \subseteq$ the closed domains of T if and only if F is closed-domains-family.

(73) For every family F of subsets of T such that F is closed-domains-family holds F is domains-family.

(74) For every family F of subsets of T such that F is closed-domains-family holds F is closed.

(75) For every family F of subsets of T such that F is domains-family holds $\overline{F}$ is closed-domains-family.

(76) Let F be a family of subsets of T. Suppose F is closed-domains-family. Then $\overline{\bigcup F}$ is closed condensed and $\overline{\mathrm{Int} \bigcap F}$ is closed condensed.

(77) Let F be a family of subsets of T. Then

(i) for every subset B of T such that $B \in F$ holds $B \subseteq \overline{\bigcup F}$, and

(ii) for every subset A of T such that A is closed condensed holds if for every subset B of T such that $B \in F$ holds $B \subseteq A$, then $\overline{\bigcup F} \subseteq A$.

(78) Let F be a family of subsets of T. Then

(i) if F is closed, then for every subset B of T such that $B \in F$ holds $\overline{\mathrm{Int} \bigcap F} \subseteq B$, and

(ii) $F = \emptyset$ or for every subset A of T such that A is closed condensed holds if for every subset B of T such that $B \in F$ holds $A \subseteq B$, then $A \subseteq \overline{\mathrm{Int} \bigcap F}$.

Let us consider T and let I_1 be a family of subsets of T. We say that I_1 is open-domains-family if and only if:

(Def. 4) For every subset A of T such that $A \in I_1$ holds A is open condensed.

We now state several propositions:

(79) For every family F of subsets of T holds $F \subseteq$ the open domains of T iff F is open-domains-family.

(80) For every family F of subsets of T such that F is open-domains-family holds F is domains-family.

(81) For every family F of subsets of T such that F is open-domains-family holds F is open.

(82) For every family F of subsets of T such that F is domains-family holds $\mathrm{Int} F$ is open-domains-family.

(83) Let F be a family of subsets of T. Suppose F is open-domains-family. Then $\mathrm{Int} \bigcap F$ is open condensed and $\mathrm{Int} \overline{\bigcup F}$ is open condensed.

(84) Let F be a family of subsets of T. Then

(i) if F is open, then for every subset B of T such that $B \in F$ holds $B \subseteq \mathrm{Int} \overline{\bigcup F}$, and

(ii) for every subset A of T such that A is open condensed holds if for every subset B of T such that $B \in F$ holds $B \subseteq A$, then $\mathrm{Int} \overline{\bigcup F} \subseteq A$.

(85) Let F be a family of subsets of T. Then

(i) for every subset B of T such that $B \in F$ holds $\mathrm{Int} \bigcap F \subseteq B$, and

(ii) $F = \emptyset$ or for every subset A of T such that A is open condensed holds if for every subset B of T such that $B \in F$ holds $A \subseteq B$, then $A \subseteq \mathrm{Int} \bigcap F$.

4. Completeness of the Lattice of Domains

In the sequel T is a non empty topological space.

The following propositions are true:

(86) The carrier of the lattice of domains of T = the domains of T.

(87) Let a, b be elements of the lattice of domains of T and A, B be elements of the domains of T. If $a = A$ and $b = B$, then $a \sqcup b = \mathrm{Int}\overline{A \cup B} \cup (A \cup B)$ and $a \sqcap b = \overline{\mathrm{Int}(A \cap B)} \cap (A \cap B)$.

(88) $\perp_{\text{the lattice of domains of } T} = \emptyset_T$ and $\top_{\text{the lattice of domains of } T} = \Omega_T$.

(89) Let a, b be elements of the lattice of domains of T and A, B be elements of the domains of T. If $a = A$ and $b = B$, then $a \sqsubseteq b$ iff $A \subseteq B$.

(90) Let X be a subset of the lattice of domains of T. Then there exists an element a of the lattice of domains of T such that $X \sqsubseteq a$ and for every element b of the lattice of domains of T such that $X \sqsubseteq b$ holds $a \sqsubseteq b$.

(91) The lattice of domains of T is complete.

(92) Let F be a family of subsets of T. Suppose F is domains-family. Let X be a subset of the lattice of domains of T. If $X = F$, then $\bigsqcup_{(\text{the lattice of domains of } T)} X = \bigcup F \cup \mathrm{Int}\overline{\bigcup F}$.

(93) Let F be a family of subsets of T. Suppose F is domains-family. Let X be a subset of the lattice of domains of T such that $X = F$. Then

(i) if $X \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } T)} X = \bigcap F \cap \overline{\mathrm{Int} \bigcap F}$, and

(ii) if $X = \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } T)} X = \Omega_T$.

5. Completeness of the Lattices of Closed Domains and Open Domains

In the sequel T is a non empty topological space.

Next we state a number of propositions:

(94) The carrier of the lattice of closed domains of T = the closed domains of T.

(95) Let a, b be elements of the lattice of closed domains of T and A, B be elements of the closed domains of T. If $a = A$ and $b = B$, then $a \sqcup b = A \cup B$ and $a \sqcap b = \overline{\mathrm{Int}(A \cap B)}$.

(96) $\perp_{\text{the lattice of closed domains of } T} = \emptyset_T$ and $\top_{\text{the lattice of closed domains of } T} = \Omega_T$.

(97) Let a, b be elements of the lattice of closed domains of T and A, B be elements of the closed domains of T. If $a = A$ and $b = B$, then $a \sqsubseteq b$ iff $A \subseteq B$.

(98) Let X be a subset of the lattice of closed domains of T. Then there exists an element a of the lattice of closed domains of T such that $X \sqsubseteq a$ and for every element b of the lattice of closed domains of T such that $X \sqsubseteq b$ holds $a \sqsubseteq b$.

(99) The lattice of closed domains of T is complete.

(100) Let F be a family of subsets of T. Suppose F is closed-domains-family. Let X be a subset of the lattice of closed domains of T. If $X = F$, then $\bigsqcup_{(\text{the lattice of closed domains of } T)} X = \overline{\bigcup F}$.

(101) Let F be a family of subsets of T. Suppose F is closed-domains-family. Let X be a subset of the lattice of closed domains of T such that $X = F$. Then

(i) if $X \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of closed domains of } T)} X = \overline{\mathrm{Int} \bigcap F}$, and

(ii) if $X = \emptyset$, then $\bigsqcap_{(\text{the lattice of closed domains of } T)} X = \Omega_T$.

(102) Let F be a family of subsets of T. Suppose F is closed-domains-family. Let X be a subset of the lattice of domains of T such that $X = F$. Then

(i) if $X \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } T)} X = \overline{\mathrm{Int} \bigcap F}$, and

(ii) if $X = \emptyset$, then $\bigsqcap_{(\text{the lattice of domains of } T)} X = \Omega_T$.

(103) The carrier of the lattice of open domains of $T =$ the open domains of T.

(104) Let a, b be elements of the lattice of open domains of T and A, B be elements of the open domains of T. If $a = A$ and $b = B$, then $a \sqcup b = \mathrm{Int}\overline{A \cup B}$ and $a \sqcap b = A \cap B$.

(105) $\perp_{\text{the lattice of open domains of } T} = \emptyset_T$ and $\top_{\text{the lattice of open domains of } T} = \Omega_T$.

(106) Let a, b be elements of the lattice of open domains of T and A, B be elements of the open domains of T. If $a = A$ and $b = B$, then $a \sqsubseteq b$ iff $A \subseteq B$.

(107) Let X be a subset of the lattice of open domains of T. Then there exists an element a of the lattice of open domains of T such that $X \sqsubseteq a$ and for every element b of the lattice of open domains of T such that $X \sqsubseteq b$ holds $a \sqsubseteq b$.

(108) The lattice of open domains of T is complete.

(109) Let F be a family of subsets of T. Suppose F is open-domains-family. Let X be a subset of the lattice of open domains of T. If $X = F$, then $\bigsqcup_{(\text{the lattice of open domains of } T)} X = \mathrm{Int}\overline{\bigcup F}$.

(110) Let F be a family of subsets of T. Suppose F is open-domains-family. Let X be a subset of the lattice of open domains of T such that $X = F$. Then

(i) if $X \neq \emptyset$, then $\bigsqcap_{(\text{the lattice of open domains of } T)} X = \mathrm{Int} \bigcap F$, and

(ii) if $X = \emptyset$, then $\bigsqcap_{(\text{the lattice of open domains of } T)} X = \Omega_T$.

(111) Let F be a family of subsets of T. Suppose F is open-domains-family. Let X be a subset of the lattice of domains of T. If $X = F$, then $\bigsqcup_{(\text{the lattice of domains of } T)} X = \mathrm{Int}\overline{\bigcup F}$.

ACKNOWLEDGMENTS

The authors would like to thank to Professors A. Trybulec and Cz. Byliński for many helpful conversations during the preparation of this paper. The authors are also very grateful to G. Bancerek for acquainting them with the MIZTEX interface system for automated typesetting and translation.

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Received July 16, 1992

Published January 2, 2004