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The Lattice of Domains of a Topological Space[1]

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Summary. Let T be a topological space and let A be a subset of T. Recall that A is said to be a *closed domain* of T if $A = \overline{\mathrm{Int} A}$ and A is said to be an *open domain* of T if $A = \mathrm{Int}\overline{A}$ (see e.g. [8], [14]). Some simple generalization of these notions is the following one. A is said to be a *domain* of T provided $\mathrm{Int}\overline{A} \subseteq A \subseteq \overline{\mathrm{Int} A}$ (see [14] and compare [7]). In this paper certain connections between these concepts are introduced and studied.

Our main results are concerned with the following well–known theorems (see e.g. [9], [1]). For a given topological space all its closed domains form a Boolean lattice, and similarly all its open domains form a Boolean lattice, too. It is proved that *all domains of a given topological space form a complemented lattice.* Moreover, it is shown that both *the lattice of open domains and the lattice of closed domains are sublattices of the lattice of all domains.* In the beginning some useful theorems about subsets of topological spaces are proved and certain properties of domains, closed domains and open domains are discussed.

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The articles [11], [5], [12], [10], [15], [2], [14], [13], [3], [4], and [6] provide the notation and terminology for this paper.

1. Preliminary Theorems on Subset of Topological Spaces

In this paper T is a non empty topological space.

We now state a number of propositions:

(1) For all subsets A, B of T holds $A \cup B = \Omega_T$ iff $A^c \subseteq B$.

(2) For all subsets A, B of T holds A misses B iff $B \subseteq A^c$.

(3) For every subset A of T holds $\overline{\mathrm{Int}\overline{A}} \subseteq \overline{A}$.

(4) For every subset A of T holds $\mathrm{Int} A \subseteq \mathrm{Int}\overline{\mathrm{Int} A}$.

(5) For every subset A of T holds $\mathrm{Int}\overline{A} = \mathrm{Int}\overline{\mathrm{Int}\overline{A}}$.

(6) For all subsets A, B of T such that A is closed or B is closed holds $\overline{\mathrm{Int} A} \cup \overline{\mathrm{Int} B} = \overline{\mathrm{Int}(A \cup B)}$.

(7) For all subsets A, B of T such that A is open or B is open holds $\mathrm{Int}\overline{A} \cap \mathrm{Int}\overline{B} = \mathrm{Int}\overline{A \cap B}$.

(8) For every subset A of T holds $\mathrm{Int}(A \cap \overline{A^c}) = \emptyset_T$.

[1]This paper was done under the supervision of Z. Karno while the author was visiting the Institute of Mathematics of Warsaw University in Białystok.

(9) For every subset A of T holds $\overline{A \cup \mathrm{Int}(A^{\mathrm{c}})} = \Omega_T$.

(10) For all subsets A, B of T holds $\mathrm{Int}\overline{A \cup (\mathrm{Int}\overline{B} \cup B)} \cup (A \cup (\mathrm{Int}\overline{B} \cup B)) = \mathrm{Int}\overline{A \cup B} \cup (A \cup B)$.

(11) For all subsets A, C of T holds $\mathrm{Int}\overline{\mathrm{Int}\overline{A} \cup A \cup C} \cup (\mathrm{Int}\overline{A} \cup A \cup C) = \mathrm{Int}\overline{A \cup C} \cup (A \cup C)$.

(12) For all subsets A, B of T holds $\overline{\mathrm{Int}(A \cap (\overline{\mathrm{Int}B} \cap B))} \cap (A \cap (\overline{\mathrm{Int}B} \cap B)) = \overline{\mathrm{Int}(A \cap B)} \cap (A \cap B)$.

(13) For all subsets A, C of T holds $\overline{\mathrm{Int}(\overline{\mathrm{Int}A} \cap A \cap C)} \cap (\overline{\mathrm{Int}A} \cap A \cap C) = \overline{\mathrm{Int}(A \cap C)} \cap (A \cap C)$.

2. Properties of Domains of Topological Spaces

In the sequel T denotes a non empty topological space.

Next we state a number of propositions:

(14) $\emptyset_T$ is condensed.

(15) Ω_T is condensed.

(16) For every subset A of T such that A is condensed holds A^{c} is condensed.

(17) Let A, B be subsets of T. Suppose A is condensed and B is condensed. Then $\mathrm{Int}\overline{A \cup B} \cup (A \cup B)$ is condensed and $\overline{\mathrm{Int}(A \cap B)} \cap (A \cap B)$ is condensed.

(18) $\emptyset_T$ is closed condensed.

(19) Ω_T is closed condensed.

(20) $\emptyset_T$ is open condensed.

(21) Ω_T is open condensed.

(22) For every subset A of T holds $\overline{\mathrm{Int}A}$ is closed condensed.

(23) For every subset A of T holds $\mathrm{Int}\overline{A}$ is open condensed.

(24) For every subset A of T such that A is condensed holds $\overline{A}$ is closed condensed.

(25) For every subset A of T such that A is condensed holds $\mathrm{Int}A$ is open condensed.

(26) For every subset A of T such that A is condensed holds $\overline{A^{\mathrm{c}}}$ is closed condensed.

(27) For every subset A of T such that A is condensed holds $\mathrm{Int}(A^{\mathrm{c}})$ is open condensed.

(28) Let A, B, C be subsets of T. Suppose A is closed condensed and B is closed condensed and C is closed condensed. Then $\overline{\mathrm{Int}(A \cap \overline{\mathrm{Int}(B \cap C)})} = \overline{\mathrm{Int}(\overline{\mathrm{Int}(A \cap B)} \cap C)}$.

(29) Let A, B, C be subsets of T. Suppose A is open condensed and B is open condensed and C is open condensed. Then $\mathrm{Int}\overline{A \cup \mathrm{Int}\overline{B \cup C}} = \mathrm{Int}\overline{\mathrm{Int}\overline{A \cup B} \cup C}$.

3. The Lattice of Domains

Let T be a topological structure. The domains of T yielding a family of subsets of T is defined as follows:

(Def. 1) The domains of $T = \{A; A$ ranges over subsets of T: A is condensed$\}$.

Let T be a non empty topological space. Observe that the domains of T is non empty.

Let T be a non empty topological space. The domains union of T yielding a binary operation on the domains of T is defined as follows:

(Def. 2) For all elements A, B of the domains of T holds (the domains union of T)$(A, B) = \mathrm{Int}\overline{A \cup B} \cup (A \cup B)$.

We introduce D-Union(T) as a synonym of the domains union of T.

Let T be a non empty topological space. The domains meet of T yielding a binary operation on the domains of T is defined as follows:

(Def. 3) For all elements A, B of the domains of T holds (the domains meet of T)$(A,B) = \overline{\mathrm{Int}(A \cap B)} \cap (A \cap B)$.

We introduce D-Meet(T) as a synonym of the domains meet of T.

We now state the proposition

(30) For every non empty topological space T holds $\langle$the domains of T, D-Union(T), D-Meet(T)$\rangle$ is a complemented lattice.

Let T be a non empty topological space. The lattice of domains of T yields a complemented lattice and is defined as follows:

(Def. 4) The lattice of domains of $T = \langle$the domains of T, the domains union of T, the domains meet of $T\rangle$.

4. The Lattice of Closed Domains

Let T be a topological structure. The closed domains of T yields a family of subsets of T and is defined as follows:

(Def. 5) The closed domains of $T = \{A; A$ ranges over subsets of T: A is closed condensed$\}$.

Let T be a non empty topological space. One can check that the closed domains of T is non empty.

One can prove the following proposition

(31) For every non empty topological space T holds the closed domains of $T \subseteq$ the domains of T.

Let T be a non empty topological space. The closed domains union of T yields a binary operation on the closed domains of T and is defined as follows:

(Def. 6) For all elements A, B of the closed domains of T holds (the closed domains union of T)$(A, B) = A \cup B$.

We introduce CLD-Union(T) as a synonym of the closed domains union of T.

Next we state the proposition

(32) For all elements A, B of the closed domains of T holds (CLD-Union(T))$(A,B) =$ (D-Union(T))(A,B).

Let T be a non empty topological space. The closed domains meet of T yields a binary operation on the closed domains of T and is defined as follows:

(Def. 7) For all elements A, B of the closed domains of T holds (the closed domains meet of T)$(A, B) = \overline{\mathrm{Int}(A \cap B)}$.

We introduce CLD-Meet(T) as a synonym of the closed domains meet of T.

Next we state two propositions:

(33) For all elements A, B of the closed domains of T holds (CLD-Meet(T))$(A,B) =$ (D-Meet(T))(A,B).

(34) For every non empty topological space T holds $\langle$the closed domains of T, CLD-Union(T), CLD-Meet(T)$\rangle$ is a Boolean lattice.

Let T be a non empty topological space. The lattice of closed domains of T yielding a Boolean lattice is defined by the condition (Def. 8).

(Def. 8) The lattice of closed domains of $T = \langle$the closed domains of T, the closed domains union of T, the closed domains meet of $T\rangle$.

5. The Lattice of Open Domains

Let T be a topological structure. The open domains of T yielding a family of subsets of T is defined as follows:

(Def. 9) The open domains of $T = \{A; A$ ranges over subsets of T: A is open condensed$\}$.

Let T be a non empty topological space. Note that the open domains of T is non empty.

The following proposition is true

(35) For every non empty topological space T holds the open domains of $T \subseteq$ the domains of T.

Let T be a non empty topological space. The open domains union of T yields a binary operation on the open domains of T and is defined by:

(Def. 10) For all elements A, B of the open domains of T holds (the open domains union of T)(A, B) $= \mathrm{Int}\overline{A \cup B}$.

We introduce OPD-Union(T) as a synonym of the open domains union of T.

The following proposition is true

(36) For all elements A, B of the open domains of T holds $(\text{OPD-Union}(T))(A, B) = (\text{D-Union}(T))(A, B)$.

Let T be a non empty topological space. The open domains meet of T yielding a binary operation on the open domains of T is defined by:

(Def. 11) For all elements A, B of the open domains of T holds (the open domains meet of T)(A, B) $= A \cap B$.

We introduce OPD-Meet(T) as a synonym of the open domains meet of T.

The following propositions are true:

(37) For all elements A, B of the open domains of T holds $(\text{OPD-Meet}(T))(A, B) = (\text{D-Meet}(T))(A, B)$.

(38) For every non empty topological space T holds $\langle$the open domains of T, OPD-Union(T), OPD-Meet(T)$\rangle$ is a Boolean lattice.

Let T be a non empty topological space. The lattice of open domains of T yielding a Boolean lattice is defined by the condition (Def. 12).

(Def. 12) The lattice of open domains of $T = \langle$the open domains of T, the open domains union of T, the open domains meet of $T\rangle$.

6. Connections between Lattices of Domains

In the sequel T denotes a non empty topological space.

We now state several propositions:

(39) $\text{CLD-Union}(T) = \text{D-Union}(T) \restriction [\!:$ the closed domains of T, the closed domains of $T :\!]$.

(40) $\text{CLD-Meet}(T) = \text{D-Meet}(T) \restriction [\!:$ the closed domains of T, the closed domains of $T :\!]$.

(41) The lattice of closed domains of T is a sublattice of the lattice of domains of T.

(42) $\text{OPD-Union}(T) = \text{D-Union}(T) \restriction [\!:$ the open domains of T, the open domains of $T :\!]$.

(43) $\text{OPD-Meet}(T) = \text{D-Meet}(T) \restriction [\!:$ the open domains of T, the open domains of $T :\!]$.

(44) The lattice of open domains of T is a sublattice of the lattice of domains of T.

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