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Lower Tolerance. Preliminaries to Wroclaw Taxonomy[1]

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Summary. The paper introduces some preliminary notions concerning the Wroclaw taxonomy according to [14]. The classifications and tolerances are defined and considered w.r.t. sets and metric spaces. We prove theorems showing various classifications based on tolerances.

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The articles [18], [8], [20], [2], [19], [7], [21], [23], [5], [22], [6], [13], [16], [12], [11], [10], [17], [3], [4], [15], [1], and [9] provide the notation and terminology for this paper.

1. Preliminaries

In this paper A, X denote non empty sets, f denotes a partial function from $[:X, X:]$ to $\mathbb{R}$, and a denotes a real number.

Let us observe that there exists a real number which is non negative.

We now state a number of propositions:

(1) For every finite sequence p and for every natural number k such that $k+1 \in \operatorname{dom} p$ and $k \notin \operatorname{dom} p$ holds $k = 0$.

(2) Let p be a finite sequence and i, j be natural numbers. Suppose $i \in \operatorname{dom} p$ and $j \in \operatorname{dom} p$ and for every natural number k such that $k \in \operatorname{dom} p$ and $k+1 \in \operatorname{dom} p$ holds $p(k) = p(k+1)$. Then $p(i) = p(j)$.

(3) For every set X and for every binary relation R on X such that R is reflexive in X holds $\operatorname{dom} R = X$.

(4) For every set X and for every binary relation R on X such that R is reflexive in X holds $\operatorname{rng} R = X$.

(5) For every set X and for every binary relation R on X such that R is reflexive in X holds R^* is reflexive in X.

(6) Let X, x, y be sets and R be a binary relation on X. Suppose R is reflexive in X. If R reduces x to y and $x \in X$, then $\langle x, y \rangle \in R^*$.

(7) Let X be a set and R be a binary relation on X. If R is reflexive in X and symmetric in X, then R^* is symmetric in X.

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(8) For every set X and for every binary relation R on X such that R is reflexive in X holds R^* is transitive in X.

(9) Let X be a non empty set and R be a binary relation on X. Suppose R is reflexive in X and symmetric in X. Then R^* is an equivalence relation of X.

(10) For all binary relations R_1, R_2 on X such that $R_1 \subseteq R_2$ holds ${R_1}^* \subseteq {R_2}^*$.

(11) SmallestPartition(A) is finer than $\{A\}$.

2. The Notion of Classification

Let A be a non empty set. A subset of PARTITIONS(A) is called a classification of A if:

(Def. 1) For all partitions X, Y of A such that $X \in$ it and $Y \in$ it holds X is finer than Y or Y is finer than X.

The following three propositions are true:

(12) $\{\{A\}\}$ is a classification of A.

(13) $\{$SmallestPartition$(A)\}$ is a classification of A.

(14) For every subset S of PARTITIONS(A) such that $S = \{\{A\},$SmallestPartition$(A)\}$ holds S is a classification of A.

Let A be a non empty set. A subset of PARTITIONS(A) is called a strong classification of A if:

(Def. 2) It is a classification of A and $\{A\} \in$ it and SmallestPartition$(A) \in$ it.

We now state the proposition

(15) For every subset S of PARTITIONS(A) such that $S = \{\{A\},$SmallestPartition$(A)\}$ holds S is a strong classification of A.

3. The Tolerance on a Non Empty Set

Let X be a non empty set, let f be a partial function from $[:X, X:]$ to $\mathbb{R}$, and let a be a real number. The functor $\mathrm{T_l}(f,a)$ yields a binary relation on X and is defined by:

(Def. 3) For all elements x, y of X holds $\langle x, y\rangle \in \mathrm{T_l}(f,a)$ iff $f(x, y) \leq a$.

One can prove the following propositions:

(16) If f is Reflexive and $a \geq 0$, then $\mathrm{T_l}(f,a)$ is reflexive in X.

(17) If f is symmetric, then $\mathrm{T_l}(f,a)$ is symmetric in X.

(18) If $a \geq 0$ and f is Reflexive and symmetric, then $\mathrm{T_l}(f,a)$ is a tolerance of X.

(19) Let X be a non empty set, f be a partial function from $[:X, X:]$ to $\mathbb{R}$, and a_1, a_2 be real numbers. If $a_1 \leq a_2$, then $\mathrm{T_l}(f,a_1) \subseteq \mathrm{T_l}(f,a_2)$.

Let X be a set and let f be a partial function from $[:X, X:]$ to $\mathbb{R}$. We say that f is non-negative if and only if:

(Def. 4) For all elements x, y of X holds $f(x, y) \geq 0$.

One can prove the following three propositions:

(20) Let X be a non empty set, f be a partial function from $[:X, X:]$ to $\mathbb{R}$, and x, y be sets. Suppose f is non-negative, Reflexive, and discernible. If $\langle x, y\rangle \in \mathrm{T_l}(f,0)$, then $x = y$.

(21) Let X be a non empty set, f be a partial function from $[:X, X:]$ to $\mathbb{R}$, and x be an element of X. If f is Reflexive and discernible, then $\langle x, x\rangle \in \mathrm{T_l}(f,0)$.

(22) Let X be a non empty set, f be a partial function from $[:X, X:]$ to $\mathbb{R}$, and a be a real number. Suppose $\mathrm{T_l}(f,a)$ is reflexive in X and f is symmetric. Then $(\mathrm{T_l}(f,a))^*$ is an equivalence relation of X.

4. The Partitions Defined by Lower Tolerance

The following propositions are true:

(23) Let X be a non empty set and f be a partial function from $[:X, X:]$ to $\mathbb{R}$. Suppose f is non-negative, Reflexive, and discernible. Then $(\mathrm{T_l}(f,0))^* = \mathrm{T_l}(f,0)$.

(24) Let X be a non empty set, f be a partial function from $[:X, X:]$ to $\mathbb{R}$, and R be an equivalence relation of X. Suppose $R = (\mathrm{T_l}(f,0))^*$ and f is non-negative, Reflexive, and discernible. Then $R = \mathrm{id}_X$.

(25) Let X be a non empty set, f be a partial function from $[:X, X:]$ to $\mathbb{R}$, and R be an equivalence relation of X. Suppose $R = (\mathrm{T_l}(f,0))^*$ and f is non-negative, Reflexive, and discernible. Then $\mathrm{Classes}\,R = \mathrm{SmallestPartition}(X)$.

(26) Let X be a finite non empty subset of $\mathbb{R}$, f be a function from $[:X, X:]$ into $\mathbb{R}$, z be a finite non empty subset of $\mathbb{R}$, and A be a real number. If $z = \mathrm{rng}\,f$ and $A \geq \max z$, then for all elements x, y of X holds $f(x, y) \leq A$.

(27) Let X be a finite non empty subset of $\mathbb{R}$, f be a function from $[:X, X:]$ into $\mathbb{R}$, z be a finite non empty subset of $\mathbb{R}$, and A be a real number. Suppose $z = \mathrm{rng}\,f$ and $A \geq \max z$. Let R be an equivalence relation of X. If $R = (\mathrm{T_l}(f,A))^*$, then $\mathrm{Classes}\,R = \{X\}$.

(28) Let X be a finite non empty subset of $\mathbb{R}$, f be a function from $[:X, X:]$ into $\mathbb{R}$, z be a finite non empty subset of $\mathbb{R}$, and A be a real number. If $z = \mathrm{rng}\,f$ and $A \geq \max z$, then $(\mathrm{T_l}(f,A))^* = \mathrm{T_l}(f,A)$.

5. The Classification on a Non Empty Set

Let X be a non empty set and let f be a partial function from $[:X, X:]$ to $\mathbb{R}$. The functor $\mathrm{FamClass}\,f$ yielding a subset of $\mathrm{PARTITIONS}(X)$ is defined by the condition (Def. 5).

(Def. 5) Let x be a set. Then $x \in \mathrm{FamClass}\,f$ if and only if there exists a non negative real number a and there exists an equivalence relation R of X such that $R = (\mathrm{T_l}(f,a))^*$ and $\mathrm{Classes}\,R = x$.

Next we state four propositions:

(29) Let X be a non empty set, f be a partial function from $[:X, X:]$ to $\mathbb{R}$, and a be a non negative real number. If $\mathrm{T_l}(f,a)$ is reflexive in X and f is symmetric, then $\mathrm{FamClass}\,f$ is a non empty set.

(30) Let X be a finite non empty subset of $\mathbb{R}$ and f be a function from $[:X, X:]$ into $\mathbb{R}$. If f is symmetric and non-negative, then $\{X\} \in \mathrm{FamClass}\,f$.

(31) For every non empty set X and for every partial function f from $[:X, X:]$ to $\mathbb{R}$ holds $\mathrm{FamClass}\,f$ is a classification of X.

(32) Let X be a finite non empty subset of $\mathbb{R}$ and f be a function from $[:X, X:]$ into $\mathbb{R}$. Suppose $\mathrm{SmallestPartition}(X) \in \mathrm{FamClass}\,f$ and f is symmetric and non-negative. Then $\mathrm{FamClass}\,f$ is a strong classification of X.

6. The Classification on a Metric Space

Let M be a metric structure, let a be a real number, and let x, y be elements of M. We say that x, y are in tolerance w.r.t. a if and only if:

(Def. 6) $\rho(x,y) \leq a$.

Let M be a non empty metric structure and let a be a real number. The functor $\mathrm{T_m}(M,a)$ yields a binary relation on M and is defined by:

(Def. 7) For all elements x, y of M holds $\langle x, y\rangle \in \mathrm{T_m}(M,a)$ iff x, y are in tolerance w.r.t. a.

One can prove the following two propositions:

(33) For every non empty metric structure M and for every real number a holds $\mathrm{T_m}(M,a) = \mathrm{T_l}(\text{the distance of } M, a)$.

(34) Let M be a non empty Reflexive symmetric metric structure, a be a real number, and T be a relation between the carrier of M and the carrier of M. If $T = \mathrm{T_m}(M,a)$ and $a \geq 0$, then T is a tolerance of the carrier of M.

Let M be a Reflexive symmetric non empty metric structure. The functor MetricFamClass M yielding a subset of PARTITIONS(the carrier of M) is defined by the condition (Def. 8).

(Def. 8) Let x be a set. Then $x \in$ MetricFamClass M if and only if there exists a non negative real number a and there exists an equivalence relation R of M such that $R = (\mathrm{T_m}(M,a))^*$ and Classes $R = x$.

Next we state several propositions:

(35) For every Reflexive symmetric non empty metric structure M holds MetricFamClass M = FamClass the distance of M.

(36) Let M be a non empty metric space and R be an equivalence relation of M. If $R = (\mathrm{T_m}(M,0))^*$, then Classes R = SmallestPartition(the carrier of M).

(37) For every Reflexive symmetric bounded non empty metric structure M such that $a \geq \varnothing(\Omega_M)$ holds $\mathrm{T_m}(M,a) = \nabla_{\text{the carrier of } M}$.

(38) For every Reflexive symmetric bounded non empty metric structure M such that $a \geq \varnothing(\Omega_M)$ holds $\mathrm{T_m}(M,a) = (\mathrm{T_m}(M,a))^*$.

(39) For every Reflexive symmetric bounded non empty metric structure M such that $a \geq \varnothing(\Omega_M)$ holds $(\mathrm{T_m}(M,a))^* = \nabla_{\text{the carrier of } M}$.

(40) Let M be a Reflexive symmetric bounded non empty metric structure, R be an equivalence relation of M, and a be a non negative real number. If $a \geq \varnothing(\Omega_M)$ and $R = (\mathrm{T_m}(M,a))^*$, then Classes $R = \{\text{the carrier of } M\}$.

Let M be a Reflexive symmetric triangle non empty metric structure and let C be a non empty bounded subset of M. Note that $\varnothing C$ is non negative.

One can prove the following propositions:

(41) For every bounded non empty metric space M holds $\{\text{the carrier of } M\} \in$ MetricFamClass M.

(42) For every Reflexive symmetric non empty metric structure M holds MetricFamClass M is a classification of the carrier of M.

(43) For every bounded non empty metric space M holds MetricFamClass M is a strong classification of the carrier of M.

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