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# $T_0$ Topological Spaces

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The articles [8], [10], [5], [13], [12], [11], [1], [3], [2], [6], [4], [9], and [7] provide the notation and terminology for this paper.

One can prove the following two propositions:

(1) Let $A$, $B$ be non empty sets and $R_1$, $R_2$ be relations between $A$ and $B$. Suppose that for every element $x$ of $A$ and for every element $y$ of $B$ holds $\langle x, y\rangle \in R_1$ iff $\langle x, y\rangle \in R_2$. Then $R_1 = R_2$.

(2) Let $X$, $Y$ be non empty sets, $f$ be a function from $X$ into $Y$, and $A$ be a subset of $X$. Suppose that for all elements $x_1$, $x_2$ of $X$ such that $x_1 \in A$ and $f(x_1) = f(x_2)$ holds $x_2 \in A$. Then $f^{-1}(f^{\circ}A) = A$.

Let $T$, $S$ be topological structures. We say that $T$ and $S$ are homeomorphic if and only if:

(Def. 1) There exists a map from $T$ into $S$ which is a homeomorphism.

Let $T$, $S$ be topological structures and let $f$ be a map from $T$ into $S$. We say that $f$ is open if and only if:

(Def. 2) For every subset $A$ of $T$ such that $A$ is open holds $f^{\circ}A$ is open.

Let $T$ be a non empty topological structure. The functor Indiscernibility($T$) yields an equivalence relation of the carrier of $T$ and is defined by the condition (Def. 3).

(Def. 3) Let $p$, $q$ be points of $T$. Then $\langle p, q\rangle \in$ Indiscernibility($T$) if and only if for every subset $A$ of $T$ such that $A$ is open holds $p \in A$ iff $q \in A$.

Let $T$ be a non empty topological structure. The functor $T_{/\,\mathrm{Indiscernibility}\,T}$ yielding a non empty partition of the carrier of $T$ is defined as follows:

(Def. 4) $T_{/\,\mathrm{Indiscernibility}\,T} = \mathrm{Classes}\,\mathrm{Indiscernibility}(T)$.

Let $T$ be a non empty topological space. The functor $T_0$-reflex($T$) yielding a topological space is defined as follows:

(Def. 5) $T_0$-reflex($T$) = the decomposition space of $T_{/\,\mathrm{Indiscernibility}\,T}$.

Let $T$ be a non empty topological space. Note that $T_0$-reflex($T$) is non empty.

Let $T$ be a non empty topological space. The functor $T_0$-map($T$) yields a continuous map from $T$ into $T_0$-reflex($T$) and is defined as follows:

(Def. 6) $T_0$-map($T$) = the projection onto $T_{/\,\mathrm{Indiscernibility}\,T}$.

We now state a number of propositions:

(3) For every non empty topological space $T$ and for every point $p$ of $T$ holds $p \in (T_0\text{-map}(T))(p)$.

(4) For every non empty topological space $T$ holds $\mathrm{dom}\, T_0\text{-map}(T) =$ the carrier of $T$ and $\mathrm{rng}\, T_0\text{-map}(T) \subseteq$ the carrier of $T_0$-reflex$(T)$.

(5) Let $T$ be a non empty topological space. Then the carrier of $T_0$-reflex$(T) = T_{/\,\mathrm{Indiscernibility}\,T}$ and the topology of $T_0$-reflex$(T) = \{A;A$ ranges over subsets of $T_{/\,\mathrm{Indiscernibility}\,T}$: $\bigcup A \in$ the topology of $T\}$.

(6) Let $T$ be a non empty topological space and $V$ be a subset of $T_0$-reflex$(T)$. Then $V$ is open if and only if $\bigcup V \in$ the topology of $T$.

(7) Let $T$ be a non empty topological space and $C$ be a set. Then $C$ is a point of $T_0$-reflex$(T)$ if and only if there exists a point $p$ of $T$ such that $C = [p]_{\mathrm{Indiscernibility}(T)}$.

(8) For every non empty topological space $T$ and for every point $p$ of $T$ holds $(T_0\text{-map}(T))(p) = [p]_{\mathrm{Indiscernibility}(T)}$.

(9) For every non empty topological space $T$ and for all points $p$, $q$ of $T$ holds $(T_0\text{-map}(T))(q) = (T_0\text{-map}(T))(p)$ iff $\langle q, p\rangle \in \mathrm{Indiscernibility}(T)$.

(10) Let $T$ be a non empty topological space and $A$ be a subset of $T$. Suppose $A$ is open. Let $p$, $q$ be points of $T$. If $p \in A$ and $(T_0\text{-map}(T))(p) = (T_0\text{-map}(T))(q)$, then $q \in A$.

(11) Let $T$ be a non empty topological space and $A$ be a subset of $T$. Suppose $A$ is open. Let $C$ be a subset of $T$. If $C \in T_{/\,\mathrm{Indiscernibility}\,T}$ and $C$ meets $A$, then $C \subseteq A$.

(12) For every non empty topological space $T$ holds $T_0$-map$(T)$ is open.

Let $I_1$ be a topological structure. We say that $I_1$ is discernible if and only if the conditions (Def. 7) are satisfied.

(Def. 7)(i) $I_1$ is empty, or

(ii) for all points $x$, $y$ of $I_1$ such that $x \neq y$ there exists a subset $V$ of $I_1$ such that $V$ is open but $x \in V$ and $y \notin V$ or $y \in V$ and $x \notin V$.

Let us note that there exists a topological space which is discernible and non empty.

A $T_0$-space is a discernible non empty topological space.

The following propositions are true:

(13) For every non empty topological space $T$ holds $T_0$-reflex$(T)$ is a $T_0$-space.

(14) Let $T$, $S$ be non empty topological spaces. Given a map $h$ from $T_0$-reflex$(S)$ into $T_0$-reflex$(T)$ such that $h$ is a homeomorphism and $T_0$-map$(T)$ and $h \cdot T_0$-map$(S)$ are fiberwise equipotent. Then $T$ and $S$ are homeomorphic.

(15) Let $T$ be a non empty topological space, $T_0$ be a $T_0$-space, $f$ be a continuous map from $T$ into $T_0$, and $p$, $q$ be points of $T$. If $\langle p, q\rangle \in \mathrm{Indiscernibility}(T)$, then $f(p) = f(q)$.

(16) Let $T$ be a non empty topological space, $T_0$ be a $T_0$-space, $f$ be a continuous map from $T$ into $T_0$, and $p$ be a point of $T$. Then $f^{\circ}([p]_{\mathrm{Indiscernibility}(T)}) = \{f(p)\}$.

(17) Let $T$ be a non empty topological space, $T_0$ be a $T_0$-space, and $f$ be a continuous map from $T$ into $T_0$. Then there exists a continuous map $h$ from $T_0$-reflex$(T)$ into $T_0$ such that $f = h \cdot T_0$-map$(T)$.

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