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Infimum and Supremum of the Set of Real Numbers. Measure Theory

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Summary. We introduce some properties of the least upper bound and the greatest lower bound of the subdomain of $\overline{\mathbb{R}}$ numbers, where $\overline{\mathbb{R}}$ denotes the enlarged set of real numbers, $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$. The paper contains definitions of majorant and minorant elements, bounded from above, bounded from below and bounded sets, sup and inf of set, for nonempty subset of $\overline{\mathbb{R}}$. We prove theorems describing the basic relationships among those definitions. The work is the first part of the series of articles concerning the Lebesgue measure theory.

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The articles [3], [2], [5], [1], and [4] provide the notation and terminology for this paper.

The set $+\infty$ is defined by:

(Def. 1) $+\infty = \mathbb{R}$.

We now state the proposition

(2)[1] $+\infty \notin \mathbb{R}$.

Let I_1 be a set. We say that I_1 is positive infinite number-like if and only if:

(Def. 2) $I_1 = +\infty$.

Let us observe that there exists a set which is positive infinite number-like.

A positive infinite number is a positive infinite number-like set.

We now state the proposition

(4)[2] $+\infty$ is a positive infinite number.

The set $-\infty$ is defined by:

(Def. 3) $-\infty = \{\mathbb{R}\}$.

Next we state the proposition

(6)[3] $-\infty \notin \mathbb{R}$.

Let I_1 be a set. We say that I_1 is negative infinite number-like if and only if:

[1] The proposition (1) has been removed.
[2] The proposition (3) has been removed.
[3] The proposition (5) has been removed.

(Def. 4) $I_1 = -\infty$.

One can verify that there exists a set which is negative infinite number-like.
A negative infinite number is a negative infinite number-like set.
One can prove the following proposition

(8)[4] $-\infty$ is a negative infinite number.

Let I_1 be a set. We say that I_1 is extended real if and only if:

(Def. 5) $I_1 \in \mathbb{R} \cup \{-\infty, +\infty\}$.

Let us note that there exists a set which is extended real.
The set $\overline{\mathbb{R}}$ is defined as follows:

(Def. 6) $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$.

Let us mention that every element of $\overline{\mathbb{R}}$ is extended real.
An extended real number is an element of $\overline{\mathbb{R}}$.
Let us note that every real number is extended real.
The following propositions are true:

(10)[5] Every real number is an extended real number.

(11) For every set x such that $x = -\infty$ or $x = +\infty$ holds x is an extended real number.

$+\infty$ is an extended real number. Then $-\infty$ is an extended real number.
The following proposition is true

(14)[6] $-\infty \neq +\infty$.

Let x, y be extended real numbers. The predicate $x \leq y$ is defined as follows:

(Def. 7)(i) There exist real numbers p, q such that $p = x$ and $q = y$ and $p \leq q$ if $x \in \mathbb{R}$ and $y \in \mathbb{R}$,

(ii) $x = -\infty$ or $y = +\infty$, otherwise.

Let us notice that the predicate $x \leq y$ is reflexive and connected. We introduce $y < x$ as an antonym of $x \leq y$.

Next we state several propositions:

(16)[7] Let x, y be extended real numbers. Suppose x is a real number and y is a real number. Then $x \leq y$ if and only if there exist real numbers p, q such that $p = x$ and $q = y$ and $p \leq q$.

(17) For every extended real number x such that $x \in \mathbb{R}$ holds $x \nleq -\infty$.

(18) For every extended real number x such that $x \in \mathbb{R}$ holds $+\infty \nleq x$.

(19) $+\infty \nleq -\infty$.

(20) For every extended real number x holds $x \leq +\infty$.

(21) For every extended real number x holds $-\infty \leq x$.

(22) For all extended real numbers x, y such that $x \leq y$ and $y \leq x$ holds $x = y$.

(23) For every extended real number x such that $x \leq -\infty$ holds $x = -\infty$.

(24) For every extended real number x such that $+\infty \leq x$ holds $x = +\infty$.

[4] The proposition (7) has been removed.
[5] The proposition (9) has been removed.
[6] The propositions (12) and (13) have been removed.
[7] The proposition (15) has been removed.

The scheme *SepR eal* concerns a unary predicate $\mathcal{P}$, and states that:

There exists a subset X of $\mathbb{R} \cup \{-\infty, +\infty\}$ such that for every extended real number x holds $x \in X$ iff $\mathcal{P}[x]$

for all values of the parameters.

We now state three propositions:

(26)[8] $\overline{\mathbb{R}}$ is a non empty set.

(27) For every set x holds x is an extended real number iff $x \in \overline{\mathbb{R}}$.

(29)[9] For all extended real numbers x, y, z such that $x \leq y$ and $y \leq z$ holds $x \leq z$.

Let us observe that $\overline{\mathbb{R}}$ is non empty.

One can prove the following proposition

(31)[10] For every extended real number x such that $x \in \mathbb{R}$ holds $-\infty < x$ and $x < +\infty$.

Let X be a non empty subset of $\overline{\mathbb{R}}$. An extended real number is said to be a majorant of X if:

(Def. 9)[11] For every extended real number x such that $x \in X$ holds $x \leq$ it.

Next we state two propositions:

(33)[12] For every non empty subset X of $\overline{\mathbb{R}}$ holds $+\infty$ is a majorant of X.

(34) Let X, Y be non empty subsets of $\overline{\mathbb{R}}$. Suppose $X \subseteq Y$. Let U_1 be an extended real number. If U_1 is a majorant of Y, then U_1 is a majorant of X.

Let X be a non empty subset of $\overline{\mathbb{R}}$. An extended real number is said to be a minorant of X if:

(Def. 10) For every extended real number x such that $x \in X$ holds it $\leq x$.

We now state two propositions:

(36)[13] For every non empty subset X of $\overline{\mathbb{R}}$ holds $-\infty$ is a minorant of X.

(39)[14] Let X, Y be non empty subsets of $\overline{\mathbb{R}}$. Suppose $X \subseteq Y$. Let L_1 be an extended real number. If L_1 is a minorant of Y, then L_1 is a minorant of X.

$\mathbb{R}$ is a non empty subset of $\overline{\mathbb{R}}$.

One can prove the following two propositions:

(41)[15] $+\infty$ is a majorant of $\mathbb{R}$.

(42) $-\infty$ is a minorant of $\mathbb{R}$.

Let X be a non empty subset of $\overline{\mathbb{R}}$. We say that X is upper bounded if and only if:

(Def. 11) There exists a majorant U_1 of X such that $U_1 \in \mathbb{R}$.

We now state two propositions:

(44)[16] For all non empty subsets X, Y of $\overline{\mathbb{R}}$ such that $X \subseteq Y$ holds if Y is upper bounded, then X is upper bounded.

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[8] The proposition (25) has been removed.
[9] The proposition (28) has been removed.
[10] The proposition (30) has been removed.
[11] The definition (Def. 8) has been removed.
[12] The proposition (32) has been removed.
[13] The proposition (35) has been removed.
[14] The propositions (37) and (38) have been removed.
[15] The proposition (40) has been removed.
[16] The proposition (43) has been removed.

(45) $\mathbb{R}$ is not upper bounded.

Let X be a non empty subset of $\overline{\mathbb{R}}$. We say that X is lower bounded if and only if:

(Def. 12) There exists a minorant L_1 of X such that $L_1 \in \mathbb{R}$.

One can prove the following two propositions:

(47)[17] For all non empty subsets X, Y of $\overline{\mathbb{R}}$ such that $X \subseteq Y$ holds if Y is lower bounded, then X is lower bounded.

(48) $\mathbb{R}$ is not lower bounded.

Let X be a non empty subset of $\overline{\mathbb{R}}$. We say that X is bounded if and only if:

(Def. 13) X is upper bounded and lower bounded.

We now state two propositions:

(50)[18] For all non empty subsets X, Y of $\overline{\mathbb{R}}$ such that $X \subseteq Y$ holds if Y is bounded, then X is bounded.

(51) Let X be a non empty subset of $\overline{\mathbb{R}}$. Then there exists a non empty subset Y of $\overline{\mathbb{R}}$ such that for every extended real number x holds $x \in Y$ if and only if x is a majorant of X.

Let X be a non empty subset of $\overline{\mathbb{R}}$. The functor $\overline{X}$ yields a subset of $\overline{\mathbb{R}}$ and is defined by:

(Def. 14) For every extended real number x holds $x \in \overline{X}$ iff x is a majorant of X.

Let X be a non empty subset of $\overline{\mathbb{R}}$. One can verify that $\overline{X}$ is non empty.
One can prove the following two propositions:

(54)[19] For all non empty subsets X, Y of $\overline{\mathbb{R}}$ such that $X \subseteq Y$ and for every extended real number x such that $x \in \overline{Y}$ holds $x \in \overline{X}$.

(55) Let X be a non empty subset of $\overline{\mathbb{R}}$. Then there exists a non empty subset Y of $\overline{\mathbb{R}}$ such that for every extended real number x holds $x \in Y$ if and only if x is a minorant of X.

Let X be a non empty subset of $\overline{\mathbb{R}}$. The functor $\underline{X}$ yields a subset of $\overline{\mathbb{R}}$ and is defined by:

(Def. 15) For every extended real number x holds $x \in \underline{X}$ iff x is a minorant of X.

Let X be a non empty subset of $\overline{\mathbb{R}}$. Note that $\underline{X}$ is non empty.
We now state a number of propositions:

(58)[20] For all non empty subsets X, Y of $\overline{\mathbb{R}}$ such that $X \subseteq Y$ and for every extended real number x such that $x \in \underline{Y}$ holds $x \in \underline{X}$.

(59) Let X be a non empty subset of $\overline{\mathbb{R}}$. Suppose X is upper bounded and $X \neq \{-\infty\}$. Then there exists a real number x such that $x \in X$ and $x \neq -\infty$.

(60) Let X be a non empty subset of $\overline{\mathbb{R}}$. Suppose X is lower bounded and $X \neq \{+\infty\}$. Then there exists a real number x such that $x \in X$ and $x \neq +\infty$.

(62)[21] Let X be a non empty subset of $\overline{\mathbb{R}}$. Suppose X is upper bounded and $X \neq \{-\infty\}$. Then there exists an extended real number U_1 such that U_1 is a majorant of X and for every extended real number y such that y is a majorant of X holds $U_1 \leq y$.

[17] The proposition (46) has been removed.
[18] The proposition (49) has been removed.
[19] The propositions (52) and (53) have been removed.
[20] The propositions (56) and (57) have been removed.
[21] The proposition (61) has been removed.

(63) Let X be a non empty subset of $\overline{\mathbb{R}}$. Suppose X is lower bounded and $X \neq \{+\infty\}$. Then there exists an extended real number L_1 such that L_1 is a minorant of X and for every extended real number y such that y is a minorant of X holds $y \leq L_1$.

(64) For every non empty subset X of $\overline{\mathbb{R}}$ such that $X = \{-\infty\}$ holds X is upper bounded.

(65) For every non empty subset X of $\overline{\mathbb{R}}$ such that $X = \{+\infty\}$ holds X is lower bounded.

(66) Let X be a non empty subset of $\overline{\mathbb{R}}$. Suppose $X = \{-\infty\}$. Then there exists an extended real number U_1 such that U_1 is a majorant of X and for every extended real number y such that y is a majorant of X holds $U_1 \leq y$.

(67) Let X be a non empty subset of $\overline{\mathbb{R}}$. Suppose $X = \{+\infty\}$. Then there exists an extended real number L_1 such that L_1 is a minorant of X and for every extended real number y such that y is a minorant of X holds $y \leq L_1$.

(68) Let X be a non empty subset of $\overline{\mathbb{R}}$. Suppose X is upper bounded. Then there exists an extended real number U_1 such that U_1 is a majorant of X and for every extended real number y such that y is a majorant of X holds $U_1 \leq y$.

(69) Let X be a non empty subset of $\overline{\mathbb{R}}$. Suppose X is lower bounded. Then there exists an extended real number L_1 such that L_1 is a minorant of X and for every extended real number y such that y is a minorant of X holds $y \leq L_1$.

(70) Let X be a non empty subset of $\overline{\mathbb{R}}$. Suppose X is not upper bounded. Let y be an extended real number. If y is a majorant of X, then $y = +\infty$.

(71) Let X be a non empty subset of $\overline{\mathbb{R}}$. Suppose X is not lower bounded. Let y be an extended real number. If y is a minorant of X, then $y = -\infty$.

(72) Let X be a non empty subset of $\overline{\mathbb{R}}$. Then there exists an extended real number U_1 such that U_1 is a majorant of X and for every extended real number y such that y is a majorant of X holds $U_1 \leq y$.

(73) Let X be a non empty subset of $\overline{\mathbb{R}}$. Then there exists an extended real number L_1 such that L_1 is a minorant of X and for every extended real number y such that y is a minorant of X holds $y \leq L_1$.

Let X be a non empty subset of $\overline{\mathbb{R}}$. The functor $\sup X$ yielding an extended real number is defined by:

(Def. 16) $\sup X$ is a majorant of X and for every extended real number y such that y is a majorant of X holds $\sup X \leq y$.

Next we state the proposition

(76)[22] For every non empty subset X of $\overline{\mathbb{R}}$ and for every extended real number x such that $x \in X$ holds $x \leq \sup X$.

Let X be a non empty subset of $\overline{\mathbb{R}}$. The functor $\inf X$ yields an extended real number and is defined as follows:

(Def. 17) $\inf X$ is a minorant of X and for every extended real number y such that y is a minorant of X holds $y \leq \inf X$.

The following propositions are true:

(79)[23] For every non empty subset X of $\overline{\mathbb{R}}$ and for every extended real number x such that $x \in X$ holds $\inf X \leq x$.

[22] The propositions (74) and (75) have been removed.
[23] The propositions (77) and (78) have been removed.

(80) For every non empty subset X of $\overline{\mathbb{R}}$ and for every majorant x of X such that $x \in X$ holds $x = \sup X$.

(81) For every non empty subset X of $\overline{\mathbb{R}}$ and for every minorant x of X such that $x \in X$ holds $x = \inf X$.

(82) For every non empty subset X of $\overline{\mathbb{R}}$ holds $\sup X = \inf \overline{X}$ and $\inf X = \sup \underline{X}$.

(83) For every non empty subset X of $\overline{\mathbb{R}}$ such that X is upper bounded and $X \neq \{-\infty\}$ holds $\sup X \in \mathbb{R}$.

(84) For every non empty subset X of $\overline{\mathbb{R}}$ such that X is lower bounded and $X \neq \{+\infty\}$ holds $\inf X \in \mathbb{R}$.

Let x be an extended real number. Then $\{x\}$ is a subset of $\overline{\mathbb{R}}$.
Let x, y be extended real numbers. Then $\{x, y\}$ is a subset of $\overline{\mathbb{R}}$.
We now state a number of propositions:

(85) For every extended real number x holds $\sup\{x\} = x$.

(86) For every extended real number x holds $\inf\{x\} = x$.

(87) $\sup\{-\infty\} = -\infty$.

(88) $\sup\{+\infty\} = +\infty$.

(89) $\inf\{-\infty\} = -\infty$.

(90) $\inf\{+\infty\} = +\infty$.

(91) For all non empty subsets X, Y of $\overline{\mathbb{R}}$ such that $X \subseteq Y$ holds $\sup X \leq \sup Y$.

(92) For all extended real numbers x, y and for every extended real number a such that $x \leq a$ and $y \leq a$ holds $\sup\{x, y\} \leq a$.

(93) For all extended real numbers x, y holds if $x \leq y$, then $\sup\{x, y\} = y$ and if $y \leq x$, then $\sup\{x, y\} = x$.

(94) For all non empty subsets X, Y of $\overline{\mathbb{R}}$ such that $X \subseteq Y$ holds $\inf Y \leq \inf X$.

(95) For all extended real numbers x, y and for every extended real number a such that $a \leq x$ and $a \leq y$ holds $a \leq \inf\{x, y\}$.

(96) For all extended real numbers x, y holds if $x \leq y$, then $\inf\{x, y\} = x$ and if $y \leq x$, then $\inf\{x, y\} = y$.

(97) Let X be a non empty subset of $\overline{\mathbb{R}}$ and x be an extended real number. If there exists an extended real number y such that $y \in X$ and $x \leq y$, then $x \leq \sup X$.

(98) Let X be a non empty subset of $\overline{\mathbb{R}}$ and x be an extended real number. If there exists an extended real number y such that $y \in X$ and $y \leq x$, then $\inf X \leq x$.

(99) Let X, Y be non empty subsets of $\overline{\mathbb{R}}$. Suppose that for every extended real number x such that $x \in X$ there exists an extended real number y such that $y \in Y$ and $x \leq y$. Then $\sup X \leq \sup Y$.

(100) Let X, Y be non empty subsets of $\overline{\mathbb{R}}$. Suppose that for every extended real number y such that $y \in Y$ there exists an extended real number x such that $x \in X$ and $x \leq y$. Then $\inf X \leq \inf Y$.

(101) Let X, Y be non empty subsets of $\overline{\mathbb{R}}$, U_2 be a majorant of X, and U_3 be a majorant of Y. Then $\sup\{U_2, U_3\}$ is a majorant of $X \cup Y$.

(102) Let X, Y be non empty subsets of $\overline{\mathbb{R}}$, L_2 be a minorant of X, and L_3 be a minorant of Y. Then $\inf\{L_2, L_3\}$ is a minorant of $X \cup Y$.

(103) Let X, Y, S be non empty subsets of $\overline{\mathbb{R}}$, U_2 be a majorant of X, and U_3 be a majorant of Y. If $S = X \cap Y$, then $\inf\{U_2, U_3\}$ is a majorant of S.

(104) Let X, Y, S be non empty subsets of $\overline{\mathbb{R}}$, L_2 be a minorant of X, and L_3 be a minorant of Y. If $S = X \cap Y$, then $\sup\{L_2, L_3\}$ is a minorant of S.

(105) For all non empty subsets X, Y of $\overline{\mathbb{R}}$ holds $\sup(X \cup Y) = \sup\{\sup X, \sup Y\}$.

(106) For all non empty subsets X, Y of $\overline{\mathbb{R}}$ holds $\inf(X \cup Y) = \inf\{\inf X, \inf Y\}$.

(107) For all non empty subsets X, Y, S of $\overline{\mathbb{R}}$ such that $S = X \cap Y$ holds $\sup S \leq \inf\{\sup X, \sup Y\}$.

(108) For all non empty subsets X, Y, S of $\overline{\mathbb{R}}$ such that $S = X \cap Y$ holds $\sup\{\inf X, \inf Y\} \leq \inf S$.

Let X be a non empty set. A set is called a non empty set of non empty subsets of X if:

(Def. 18) It is a non empty subset of 2^X and for every set A such that $A \in$ it holds A is a non empty set.

Let F be a non empty set of non empty subsets of $\overline{\mathbb{R}}$. The functor $\sup_{\overline{\mathbb{R}}} F$ yielding a subset of $\overline{\mathbb{R}}$ is defined as follows:

(Def. 19) For every extended real number a holds $a \in \sup_{\overline{\mathbb{R}}} F$ iff there exists a non empty subset A of $\overline{\mathbb{R}}$ such that $A \in F$ and $a = \sup A$.

Let F be a non empty set of non empty subsets of $\overline{\mathbb{R}}$. Observe that $\sup_{\overline{\mathbb{R}}} F$ is non empty.
One can prove the following three propositions:

(112)[24] Let F be a non empty set of non empty subsets of $\overline{\mathbb{R}}$ and S be a non empty subset of $\overline{\mathbb{R}}$. If $S = \bigcup F$, then $\sup S$ is a majorant of $\sup_{\overline{\mathbb{R}}} F$.

(113) Let F be a non empty set of non empty subsets of $\overline{\mathbb{R}}$ and S be a non empty subset of $\overline{\mathbb{R}}$. If $S = \bigcup F$, then $\sup \sup_{\overline{\mathbb{R}}} F$ is a majorant of S.

(114) Let F be a non empty set of non empty subsets of $\overline{\mathbb{R}}$ and S be a non empty subset of $\overline{\mathbb{R}}$. If $S = \bigcup F$, then $\sup S = \sup \sup_{\overline{\mathbb{R}}} F$.

Let F be a non empty set of non empty subsets of $\overline{\mathbb{R}}$. The functor $\inf_{\overline{\mathbb{R}}} F$ yields a subset of $\overline{\mathbb{R}}$ and is defined by:

(Def. 20) For every extended real number a holds $a \in \inf_{\overline{\mathbb{R}}} F$ iff there exists a non empty subset A of $\overline{\mathbb{R}}$ such that $A \in F$ and $a = \inf A$.

Let F be a non empty set of non empty subsets of $\overline{\mathbb{R}}$. Note that $\inf_{\overline{\mathbb{R}}} F$ is non empty.
Next we state three propositions:

(117)[25] Let F be a non empty set of non empty subsets of $\overline{\mathbb{R}}$ and S be a non empty subset of $\overline{\mathbb{R}}$. If $S = \bigcup F$, then $\inf S$ is a minorant of $\inf_{\overline{\mathbb{R}}} F$.

(118) Let F be a non empty set of non empty subsets of $\overline{\mathbb{R}}$ and S be a non empty subset of $\overline{\mathbb{R}}$. If $S = \bigcup F$, then $\inf \inf_{\overline{\mathbb{R}}} F$ is a minorant of S.

(119) Let F be a non empty set of non empty subsets of $\overline{\mathbb{R}}$ and S be a non empty subset of $\overline{\mathbb{R}}$. If $S = \bigcup F$, then $\inf S = \inf \inf_{\overline{\mathbb{R}}} F$.

[24] The propositions (109)–(111) have been removed.
[25] The propositions (115) and (116) have been removed.

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